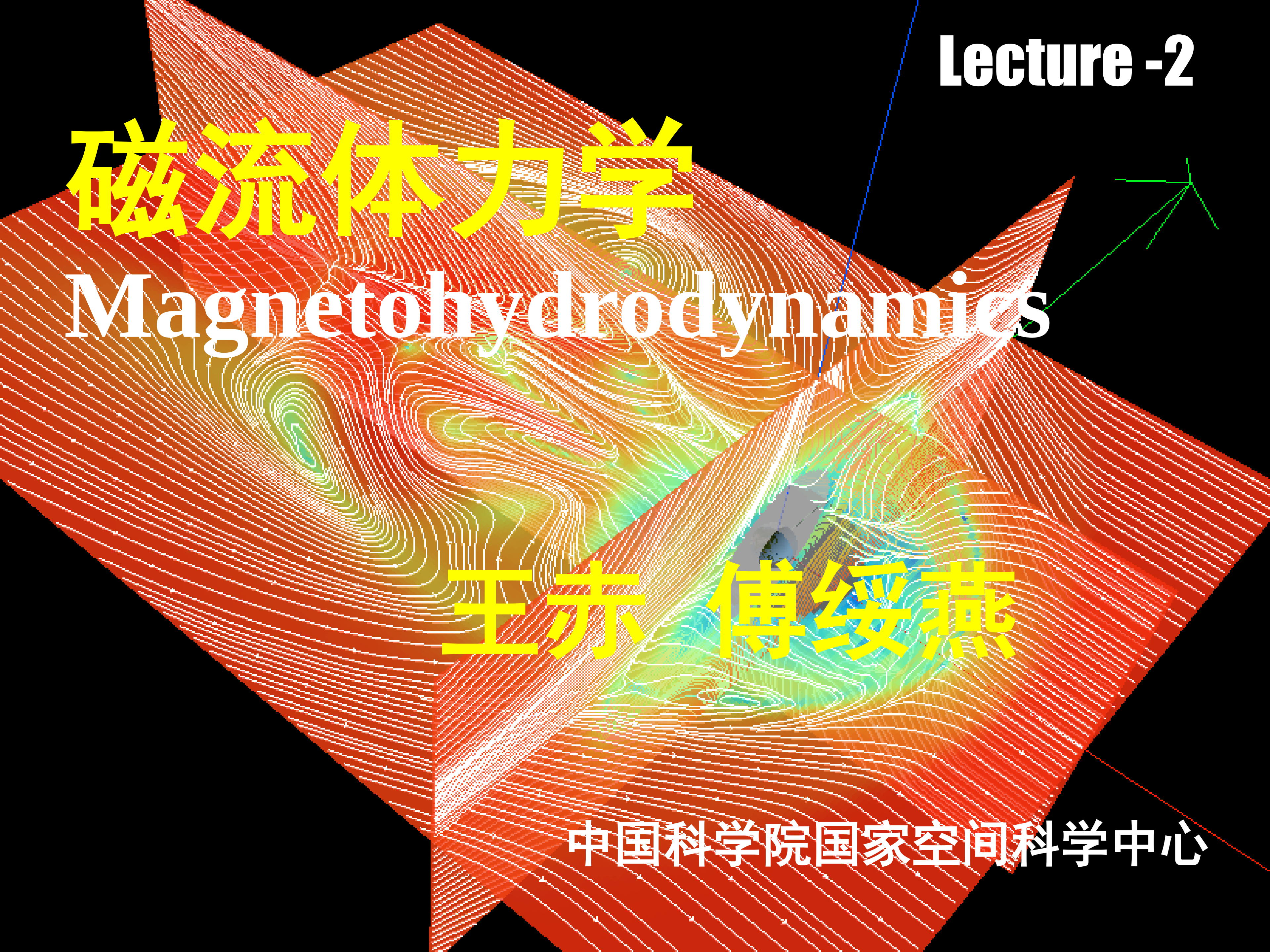

Lecture -2

磁流体力学

Magnetohydrodynamics

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General Properties of MHD Equilibria

- Equilibrium
- Plasma Beta
- Pressure and Tension
- Diamagnetic current
- Plasma Diffusion

The momentum Eq. $\rho_m(\frac{\partial}{\partial t} + \mathbf{V} \bullet \nabla)\mathbf{V} = \rho_q \mathbf{E} + \mathrm{J} \times \mathbf{B} - \nabla p$

Assuming: $\frac{\partial}{\partial t} = 0$ $\quad V = 0$ $\quad$ Ignore $\rho_q E$

$$0 = -\nabla p + \mathbf{J} \times \mathbf{B} = -\nabla p + \frac{1}{\mu_0}(\nabla \times \mathbf{B}) \times \mathbf{B} = -\nabla p + \frac{1}{\mu_0}(\mathbf{B} \bullet \nabla)\mathbf{B} - \nabla(\frac{B^2}{2\mu_0})$$

$$\nabla p = \mathbf{J} \times \mathbf{B} = \frac{1}{\mu_0}\left[(\mathbf{B} \cdot \nabla)\mathbf{B} - \frac{1}{2}\nabla B^2\right]$$

See next page

$$\nabla\left(p + \frac{B^2}{2\mu_0}\right) = \frac{1}{\mu_0}(\mathbf{B} \cdot \nabla)\mathbf{B}$$

plasma magnetohydrodynamic equilibrium

•Equilibrium structures (no time dependence, no plasma flow) are important and often approximately a reasonable assumption for space plasmas during quiet times.

Auxiliary Derivation

- Here show $(\nabla \times \mathbf{B}) \times \mathbf{B} = -\frac{1}{2}\nabla B^2 + (\mathbf{B}\cdot\nabla)\mathbf{B}$

- Consider the component form of the second term on the right:

$$(\mathbf{B}\cdot\nabla)\mathbf{B} = \left[B_x\frac{\partial}{\partial x} + B_y\frac{\partial}{\partial y} + B_z\frac{\partial}{\partial z}\right]\mathbf{B} = \hat{\mathbf{x}}\left[B_x\frac{\partial B_x}{\partial x} + B_y\frac{\partial B_x}{\partial y} + B_z\frac{\partial B_x}{\partial z}\right] + \hat{\mathbf{y}}(\cdot) + \hat{\mathbf{z}}(\cdot)$$

while the component form of the ∇B^2 term on the right is: $\nabla B^2 = \nabla(B_x^2 + B_y^2 + B_z^2) = \hat{\mathbf{x}}\left[\frac{\partial B_x^2}{\partial x} + \frac{\partial B_y^2}{\partial x} + \frac{\partial B_z^2}{\partial x}\right] + \hat{\mathbf{y}}(\cdot) + \hat{\mathbf{z}}(\cdot)$

- We now examine the component form of the left hand side:

$$(\nabla \times \mathbf{B}) \times \mathbf{B} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} & \frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x} & \frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} \\ B_x & B_y & B_z \end{vmatrix}$$

$$= \hat{\mathbf{x}}\left[\underbrace{B_x\frac{\partial B_x}{\partial x} + B_y\frac{\partial B_x}{\partial y} + B_z\frac{\partial B_x}{\partial z}}_{[(\mathbf{B}\cdot\nabla)\mathbf{B}]_x} - B_x\frac{\partial B_x}{\partial x} - B_y\frac{\partial B_y}{\partial x} - B_z\frac{\partial B_z}{\partial x}\right] + \cdots$$

$$= \hat{\mathbf{x}}\left\{\left[(\mathbf{B}\cdot\nabla)\mathbf{B}\right]_x - \frac{1}{2}\frac{\partial B_x^2}{\partial x} - \frac{1}{2}\frac{\partial B_y^2}{\partial x} - \frac{1}{2}\frac{\partial B_z^2}{\partial x}\right\} + \hat{\mathbf{y}}(\cdot) + \hat{\mathbf{z}}(\cdot)$$

$$(\nabla \times \mathbf{B}) \times \mathbf{B} = \hat{\mathbf{x}}\left\{-\frac{1}{2}\left[\nabla B^2\right]_x + \left[(\mathbf{B}\cdot\nabla)\mathbf{B}\right]_x\right\} + \hat{\mathbf{y}}(\cdot) + \hat{\mathbf{z}}(\cdot) \qquad \text{Q.E.D}$$

$$\nabla\left(p+\frac{B^2}{2\mu_0}\right)=\frac{1}{\mu_0}(\mathbf{B}\cdot\nabla)\mathbf{B}$$

- The RHS can be neglected in many cases like a straight magnetic field or when B varies slowly along B itself

The condition

$$\nabla\left(p+\frac{B^2}{2\mu_0}\right)=0 \quad \Rightarrow \quad p+\frac{B^2}{2\mu_0}=\text{constant}$$

sum of the thermal pressure and the magnetic pressure

- The ratio between the thermal (kinetic) pressure and the magnetic pressure:

Plasma β

$$\beta=\frac{2\mu_0 p}{B^2}=\frac{2\mu_0\sum nk_BT}{B^2}$$

Example : pressure-balanced plasma column θ – pinch
So called because currents flow in θ direction

Take the column to be ∞ length, uniform: B has only z – component, j has only θ component, ∇p has only r component, so we only need force: $(\mathbf{j} \times \mathbf{B})_r - (\nabla p)_r = 0$

Ampere $(\nabla \times B)_\theta = (\mu_0 \mathbf{j})_\theta$

$$j_\theta B_z - \frac{\partial}{\partial r} p = 0$$

$$-\frac{\partial}{\partial r} B_z = \mu_0 j_\theta$$

Eliminate j ⇒

$$-\frac{B_z}{\mu_0}\frac{\partial B_z}{\partial r} - \frac{\partial p}{\partial r} = 0$$

⇒

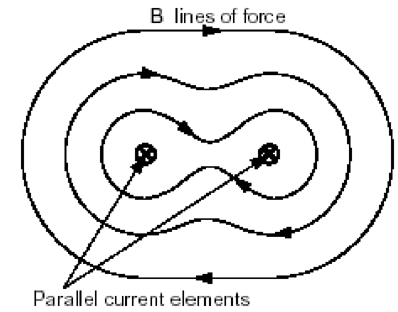
$$\frac{\partial}{\partial r}\left(\frac{B_z^2}{2\mu_0} + p\right) = 0$$

Solution:

$$\frac{B_z^2}{2\mu_0} + p = const$$

Magnetic Tension

It is know that two wires carrying parallel currents attract as if the magnetic field lines of force were under tension

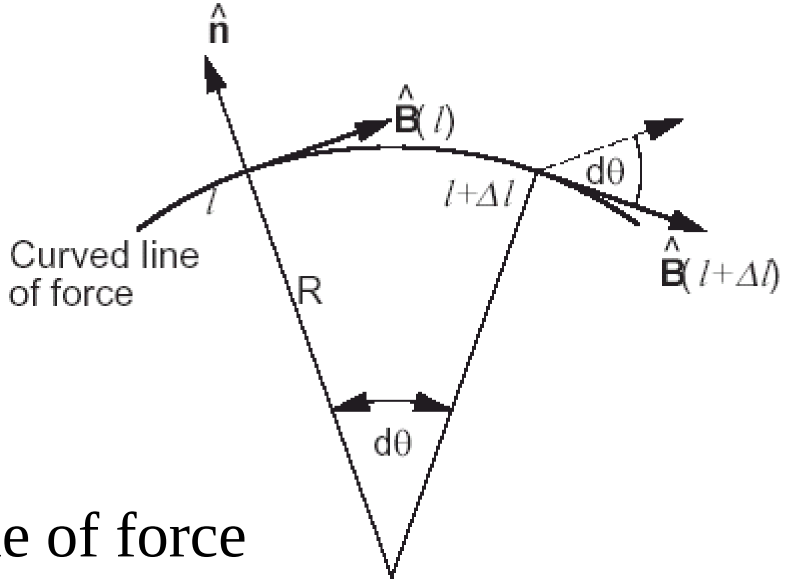


Magnetic tension is described by the term

$$(\mathbf{B} \cdot \nabla)\mathbf{B} = B_x \frac{\partial}{\partial x}(B_x\mathbf{i} + B_y\mathbf{j} + B_z\mathbf{k}) + \ldots$$

If the magnetic lines of force are straight and parallel then $B = B_x\mathbf{i}$ and $\boldsymbol{B} \cdot \nabla \boldsymbol{B} = 0$. This term is only important if the magnetic lines of force are curved.

To show this, consider the geometric construction as shown and let $\hat{\mathbf{B}} = \mathbf{B}/|\mathbf{B}|$ be the unit vector in the direction of the field. By definition:



$$\hat{\mathbf{B}} \cdot \nabla \hat{\mathbf{B}} = \frac{\partial \hat{\mathbf{B}}}{\partial l}$$

Where l is the coordinate alone the line of force

$$\frac{\Delta \hat{\mathbf{B}}}{\Delta l} = \frac{\hat{\mathbf{B}}(l + \Delta l) - \hat{\mathbf{B}}(l)}{\Delta l}$$

It is clear that $\hat{\mathbf{B}}(l + \Delta l) - \hat{\mathbf{B}}(l) = -\hat{\mathbf{n}} d\theta$

Where $\hat{\mathbf{n}}$ is the normal to the field line, while $\Delta l = R d\theta$, therefore,

$$\hat{\mathbf{B}} \cdot \nabla \hat{\mathbf{B}} = -\frac{\hat{\mathbf{n}}}{R}$$

So that the magnetic tension is in inversely proportional to the radius of curvature of the magnetic field line. The lines of force can be regarded as elastic cords under tension B^2/μ_0

Magnetic stress tensor

The existence of magnetic pressure and tension shows that the magnetic force is different in different directions, and so the magnetic force ought to be characterized by an anisotropic stress tensor.

$$\boldsymbol{J} \times \boldsymbol{B} = \frac{1}{\mu_0}(\boldsymbol{\nabla} \times \boldsymbol{B}) \times \boldsymbol{B} = \frac{1}{\mu_0}\left[-\nabla\left(\frac{B^2}{2}\right) + \boldsymbol{B} \cdot \nabla\boldsymbol{B}\right] = -\frac{1}{\mu_0}\nabla \cdot [\frac{B^2}{2}\mathrm{I} - \boldsymbol{BB}]$$

Where I is the unit tensor and the relation $\nabla \cdot (\boldsymbol{BB}) = (\boldsymbol{\nabla} \cdot \boldsymbol{B})\boldsymbol{B} + \boldsymbol{B} \cdot \boldsymbol{\nabla B} = \boldsymbol{B} \cdot \boldsymbol{\nabla B}$

$$\rho\left[\frac{\partial V}{\partial t} + V \cdot \nabla V\right] = \boldsymbol{J} \times \boldsymbol{B} - \nabla p = -[\begin{matrix} p + \frac{B^2}{2\mu_0} & & \\ & p + \frac{B^2}{2\mu_0} & \\ & & p - \frac{B^2}{2\mu_0} \end{matrix}]$$

Showing again the magnetic field acts like a pressure in the directions transverse to B (i.e. x, y directions) and like a tension the direction parallel to B.

Magnetic stress tensor

While the above interpretation is certainly useful, it can be somewhat misleading because it might be interpreted as implying the existence of a force in the direction B when in fact no such force exists because $\boldsymbol{J} \times \boldsymbol{B}$ clearly does not have a component in the B direction. A more accurate way is as follows:

Let $\boldsymbol{B} = B\boldsymbol{s}$ where $\boldsymbol{s}$ is the unit vector along the B field. $(B \cdot \nabla)B = B\frac{d(B\boldsymbol{s})}{ds} = B\frac{dB}{ds}\boldsymbol{s} + \boldsymbol{B^2}\frac{\boldsymbol{ds}}{\boldsymbol{ds}} = \frac{d(B^2/2)}{ds}\boldsymbol{s} + B^2\frac{\boldsymbol{n}}{\boldsymbol{R}}$, where n is the principle normal to the magnetic field and R is the radius of curvature

$$\boldsymbol{J} \times \boldsymbol{B} = \frac{1}{\mu_0}\left[-\nabla\left(\frac{B^2}{2}\right) + \boldsymbol{B} \cdot \nabla\boldsymbol{B}\right] = -\frac{B^2}{\mu_0}\frac{\boldsymbol{n}}{R} - \frac{1}{\mu_0}\left[\nabla\left(\frac{B^2}{2}\right) - \frac{d(B^2/2)}{ds}\boldsymbol{s}\right]$$

$$= \frac{1}{\mu_0}\left[-\nabla_\perp\left(\frac{B^2}{2}\right) - B^2\frac{\boldsymbol{n}}{R}\right]$$

The first term portrays a magnetic force due to pressure gradients perpendicular to the magnetic field. The second term describes a force which tends to straighten out magnetic curvature.

Magnetohydrostatics

- In the study of dynamical systems it is always useful to start with a study of the simplest solutions. These are usually the stationary states.
- Many physical processes in plasma systems occur slowly, i.e. on time-scales which are much longer than the typical time-scale of the system.

➢ No time dependence and no plasma flows

➢ More precise: The dynamic terms in MHD are small compared with static forces (Lorentz-force, plasma pressure gradient)

$$-\nabla p + \mathbf{J} \times \mathbf{B} = 0$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$$

$$\nabla \bullet \mathbf{B} = 0$$

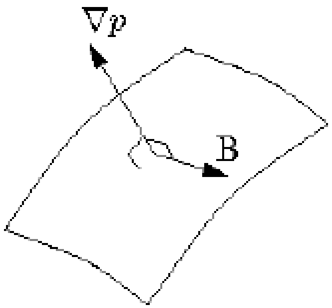
Magnetic Surface

$$0 = \mathbf{B} \cdot (-\nabla p + \mathbf{J} \times \mathbf{B}) = -\mathbf{B} \cdot \nabla p$$

Pressure is constant on a field-line (in MHD situation)

Consider some arbitrary volume in which $\nabla p \neq 0$ .Drawn contours (surface in 3-D) on which p=const. At any point on this surface, ∇p is perp to the surface.

$$-\mathbf{B} \cdot \nabla p = 0$$



Implies B is also perp to the ∇p

B lines is in the surface of p = const. In equilibrium isobaric surface are "magnetic surface"

Current Surface

$$0 = \mathbf{J} \cdot (-\nabla p + \mathbf{J} \times \mathbf{B}) = -\mathbf{J} \cdot \nabla p$$

$$\mathrm{J} \cdot \nabla p = 0$$

Isobaric Surface are “current surface”. Therefore, “Magnetic Surfaces” are “ Current Surfaces”

It is important to note that the existent of magnetic surface is guaranteed only in the MHD approximation when $\nabla p \neq 0$.

Taking account of corrections to MHD, we may not have magnetic surfaces even if $\nabla p \neq 0$

Low-beta equilibria: Force-free plasmas

In the cases the ratio of kinetic to magnetic pressure is small, $\beta \ll 1$, and we can approximately ignore ∇p. Such a equilibrium is called “force free”

$$-\nabla p + \mathbf{J} \times \mathbf{B} = 0 \quad \Rightarrow \quad \mathbf{J} \times \mathbf{B} = 0$$

Implies J and B are parallel.

$$\mathbf{J} = \mu(r)\mathbf{B}$$

Scalar function

Where μ is a scalar, which in principle can be a function of space. Current flows along field lines, but do not across. Take divergence:

$$0 = \nabla \cdot \mathbf{J} = \nabla \cdot (\mu(r)\mathbf{B}) = \mu(r)\nabla \cdot \mathbf{B} + (\mathbf{B} \cdot \nabla)\mu(r) = (\mathbf{B} \cdot \nabla)\mu(r)$$

This means that μ cannot vary along a magnetic field line. In general, μ can have different values on different field lines, but it has to be a constant on one field line.

Low-beta equilibria: Force-free plasmas

The simplest case is to consider μ to be constant. – linear force-free field.

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j} = \mu_0 \mu \mathbf{B}$$

This is a somewhat more convenient form because it is linear in B (for specified μ).

A linear equation can in general be solved by a series expansion. Since it is still a vector equation rather than a scalar equation, obtaining a general solution by series expansion is slightly complicated. Here we shall not discuss this general solution, but only consider the solution with cylindrical symmetry. Written in cylindrical coordinates assuming cylindrical symmetry (i.e. no variation of any quantity in θ or z directions):

$$
\begin{aligned}
-\frac{dB_z}{dr} &= \mu B_\theta \\
\frac{1}{r}\frac{d}{dr}(rB_\theta) &= \mu B_z
\end{aligned}
$$

Leads to a Bessel function solution:

$$
\begin{aligned}
B_z &= B_0 J_0(\mu r) \\
B_\theta &= B_0 J_1(\mu r)
\end{aligned}
$$

Where J_0 and J_1 are Bessel functions of order 0, 1.

Diamagnetic Drift

Since a fluid element is composed of many individual particles, one would expect the fluid to have drifts perpendicular to B if the individual guiding centers have such drifts. However, since the ∇p term appears only in the fluid equations, there is a drift associated with it which the fluid elements have but the particles do not have.

- Fluid momentum equation:

$$nm\left[\frac{\partial \mathbf{u}}{\partial t}+(\mathbf{u}\cdot\nabla)\mathbf{u}\right]=qn(\mathbf{E}+\mathbf{u}\times\mathbf{B})-\nabla p$$

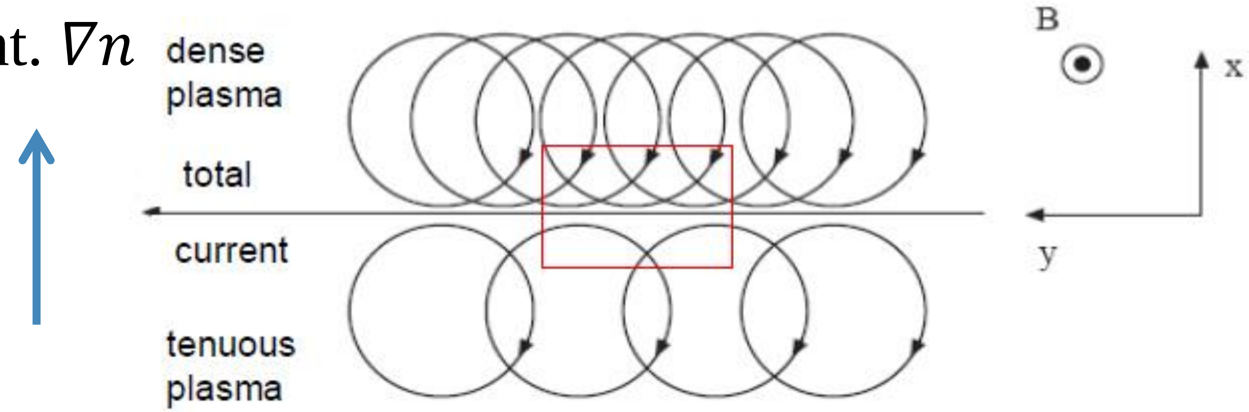
- Assuming : Uniform E and B, n and p have a gradient
- To study the motion perpendicular to B the cross product of the momentum equation with B is taken (neglecting the LHS):

$$0 = qn(\mathbf{E} + \mathbf{u} \times \mathbf{B}) \times \mathbf{B} - \nabla p \times \mathbf{B}$$

that yields

$$\mathbf{u} = \mathbf{v}_\perp = \frac{\mathbf{E} \times \mathbf{B}}{B^2} - \frac{\nabla p \times \mathbf{B}}{qnB^2}$$

- The first term is the usual ExB drift, as in the particle description, the second term is called diamagnetic drift density gradient. ∇n



- ➢ Diamagnetic current arise from Lamor motion when there is a density gradient.

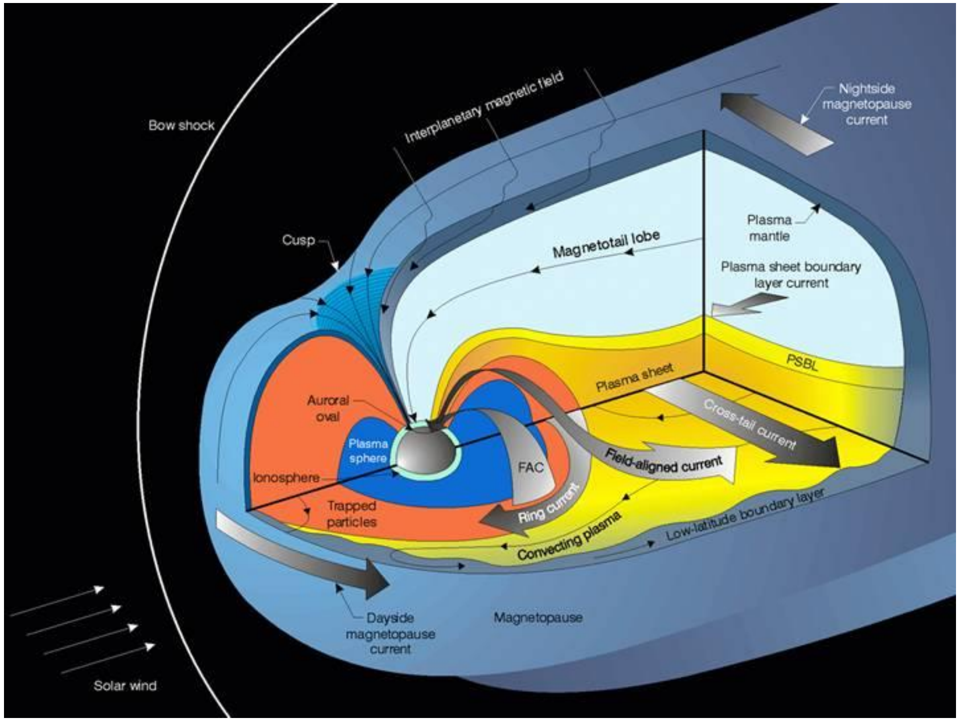
- The diamagnetic drift does not depend on the mass but changes sign with the charge: this causes a diamagnetic current since electrons and ions drift in opposite directions

$$\mathbf{j}_{dia} = \frac{B \times \nabla_{\perp} p_{\perp}}{B^2}$$

Neutral sheet current

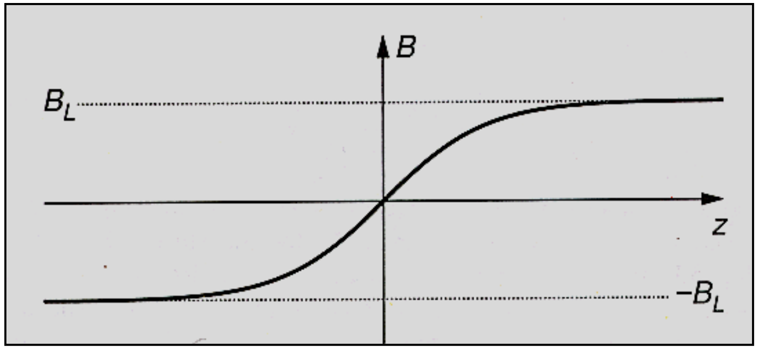
A typical example of a diamagnetic current is the neutral sheet in the magnetotail of the Earth, which divides the regions of inward (in the northern lobe) and outward magnetic fields.

Parameters:
temperature 1-10 keV,
transverse field 1-5 nT,
density 1 cm^{-3},
thickness 1-2 R_E,
very high plasma beta,
β = 10~100.



The Harris model sheet is shown below.

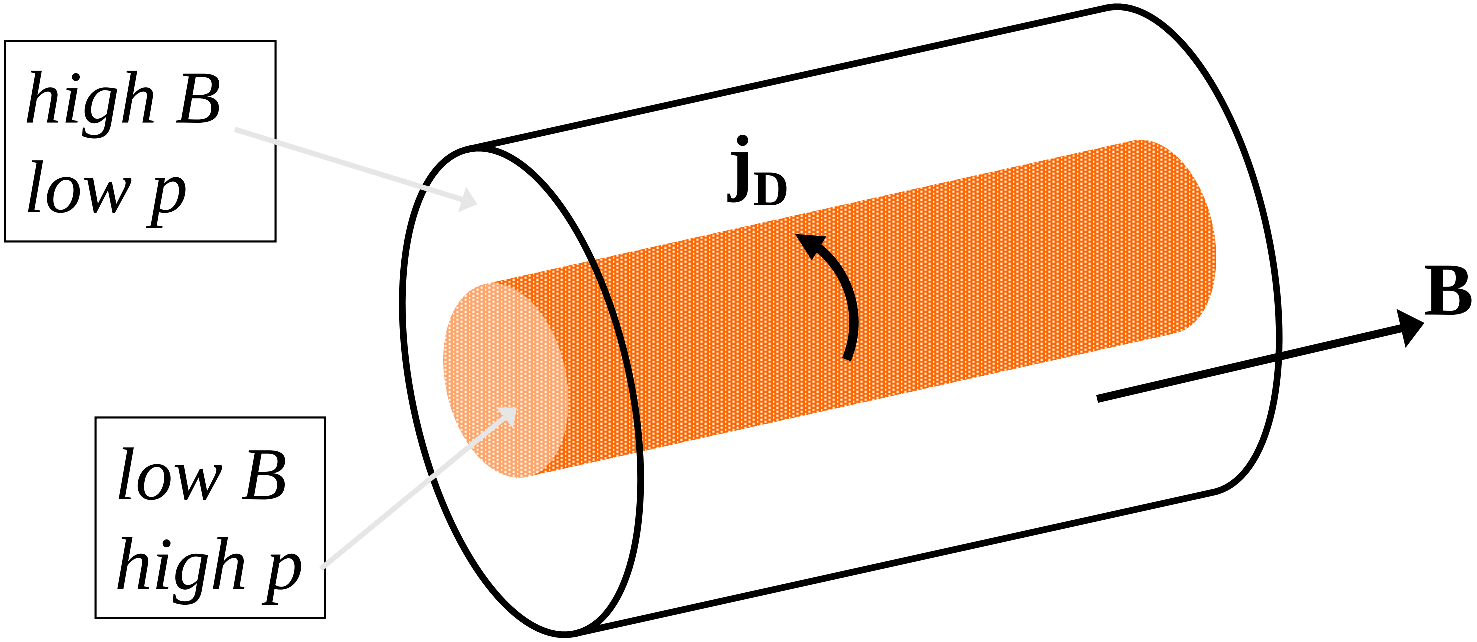
$$B_x = B_L \tanh(z/L_B)$$



Here B_L is the lobe magnetic field, and L_B its variation scale length.

The boundary of the plasma sheet is determined by a balance between the magnetic pressure of the tail lobes and the kinetic pressure of the plasma sheet plasma:

$$nkT \sim \frac{B_L^2}{2\mu_0}$$



- The diamagnetic current j_D decreases the magnetic field inside the plasma keeping the sum of thermal and magnetic pressures constant everywhere in the cylinder

Plasma Diffusion

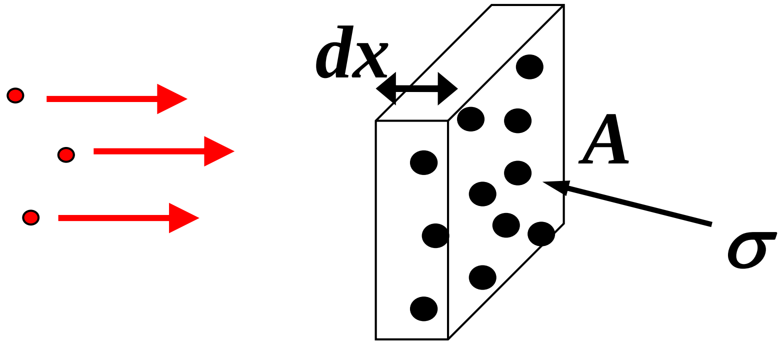
- The infinite, homogeneous plasmas for the equilibrium conditions are, of course, highly idealized.

- Plasmas follow density gradients and diffuse trying to fill lowest density regions: plasma diffusion occur, at different rates, with and without magnetic fields

- In the case of weakly ionized plasmas the diffusion occurs mainly because collisions between charged particles and neutrals

Collision Primer

- A flux of test particles is colliding with target particles of density n_T
- Target particles offer a cross-sectional area *s* and are contained in a slab of area A and thickness *dx*
- Collisions occur only when a test particle is intercepted by a target and in that case the test particle will loose all its momentum



- The number of particles in the slab is *nAdx* and the fraction area blocked by the target particles is $\frac{\sigma n A dx}{A} = \sigma n dx$
- If flux of incident particles is Γ the flux emerging from the slab is $\Gamma^* = \Gamma(1-\sigma n dx)$.

$$\frac{d\Gamma}{dx} = -n\sigma\Gamma$$

that has the solution

$$\Gamma = \Gamma_0 \exp(-n\sigma x) = \Gamma_0 \exp(-x/\lambda_m)$$

where the quantity $\lambda_m = 1/ns$ is called the mean free path for collisions

- If u is the velocity of the incident particles, the mean time between collisions is $t=\lambda_m/u$ and the mean collision frequency will be $v= u / \lambda_m=uns$

- For incident particles with a velocity distribution the collision frequency is defined by taking the average of v over that distribution

- σ can also be function of the velocity

- Consider the basic plasma diffusion process with a scalar pressure term and collisional plasma (No magnetic field)

$$nm\left[\frac{\partial \mathbf{u}}{\partial t}+(\mathbf{u}\cdot\nabla)\mathbf{u}\right]=qn\mathbf{E}-\nabla p-mn\nu\mathbf{u}$$

- For sufficiently slow motion (compared to the collision time), a steady state is considered.

$$0=qn\mathbf{E}-\nabla p-mn\nu\mathbf{u}$$

$$0 = qn\mathbf{E} - \nabla p - mn\nu\mathbf{u}$$

$$\mathbf{u} = \frac{1}{mn\nu}\left(qn\mathbf{E} - \nabla p\right)$$

- For isothermal plasmas (and subject to the ideal gas equation of state) it can be written

$$\mathbf{u} = \frac{1}{mn\nu}\left(qn\mathbf{E} - k_B T \nabla n\right) = \frac{q}{m\nu}\mathbf{E} - \frac{k_B T}{m\nu}\frac{\nabla n}{n}$$

- The coefficients of ***E*** and *grad n/n* are called mobility and diffusion coefficient

$$\mu = \frac{|q|}{m\nu} \qquad D = \frac{k_B T}{m\nu}$$

and are connected by the Einstein relation

$$\mu = \frac{|q| D}{k_B T}$$

- Introducing the mobility and the diffusion coefficient the equation for **u** becomes

$$\mathbf{u} = \frac{q}{|q|}\mu\mathbf{E} - D\frac{\nabla n}{n}$$

- The flux ***nu*** of particles can be written then as

$$\Gamma = n\mathbf{u} = \frac{q}{|q|}\mu n\mathbf{E} - D\nabla n$$

- When the particles are neutral (or E=0), the Fick's law is found as a special case :

$$\Gamma = n\mathbf{u} = -D\nabla n$$

- Fick's law describes a random-walk type of diffusion: the motion along *grad n* occurs only because there are more particles in regions with larger *n*

Diffusion Equation

- The continuity equation for each species $j=i,e$ can be written as

$$\frac{\partial n_j}{\partial t} + \nabla \cdot (n_j \mathbf{u}) = 0$$

and by using Fick's law $\Gamma_j = -D_j \nabla n_j$

$$\frac{\partial n_j}{\partial t} + \nabla \cdot \left(-D_j \nabla n_j\right) = 0$$

and for uniform diffusion coefficient yields a diffusion equation

$$\frac{\partial n_j}{\partial t} - D_j \nabla^2 n_j = 0$$

Ambipolar Diffusion

- In a bounded plasma there will be a flux of particles (diffusion) towards the container walls. Electrons and ions near the wall recombine. Plasma density towards the wall tends to zero.
- For slow flows the time dependence in the momentum equation can be neglected.
- If the electron and ion fluxes are not equal a charge unbalance will occur near the wall. This charge unbalance will eventually adjust the fluxes to maintain plasma quasi-neutrality
- Electrons are lighter and in thermodynamic equilibrium will travel faster. Electrons will be the first to leave the plasma and will establish a negative charge near the wall

- Further flow of electrons will be prevented by this negative space charge. The ion flux will be increased.
- A balance is reached when the space charge electric field produces equal ion and electron fluxes:

$$\Gamma_i = \frac{q_i}{|q_i|}\mu_i n_i \mathbf{E} - D_i \nabla n_i = \Gamma_e = \frac{q_e}{|q_e|}\mu_e n_e \mathbf{E} - D_e \nabla n_e$$

$$\mathbf{E} = \frac{D_i - D_e}{\mu_i + \mu_e}\frac{\nabla n}{n}$$

- The equilibrium flux will be then

$$\Gamma_i = \mu_i \frac{D_i - D_e}{\mu_i + \mu_e}\nabla n - D_i \nabla n_i = -\frac{\mu_i D_i + \mu_e D_e}{\mu_i + \mu_e}\nabla n$$

- ambipolar diffusion coefficient

$$D_a = \frac{\mu_i D_e + \mu_e D_i}{\mu_i + \mu_e}$$

ambipolar diffusion equation

$$\frac{\partial n}{\partial t} - D_a \nabla^2 n = 0$$

the case of $m_e << m_i$, $\mu_e \gg \mu_i$ and $T_i = T_e$ $\quad D_a \approx D_i + \frac{T_e}{T_i} D_e = 2D_i$

- The ambipolar diffusion enhances the diffusion rate by a factor of two but the diffusion is still controlled by the slower species

Collisions in Fully Ionized Plasmas

- Collisions among particles of the same species produce on average a small diffusion effect because the guiding centers remain, for the most part, in the same position.

- Collisions between particles of opposite charge can cause instead a significant change in the guiding center position: these collisions generate diffusion.

- Electrons execute a random-walk type of diffusion, ions are diffusing as a result of the cumulative effect of the collisions.

Plasma Resistivity

- The fluid equations of motion for electron and ions in presence of charged-particle collisions are

$$n_i m_i \frac{d\mathbf{u}_i}{dt} = q_i n_i \left(\mathbf{E} + \mathbf{u}_i \times \mathbf{B}\right) - \nabla \cdot \mathrm{P}_i + \mathrm{F}_{ie}$$

$$n_e m_e \frac{d\mathbf{u}_e}{dt} = q_e n_e \left(\mathbf{E} + \mathbf{u}_e \times \mathbf{B}\right) - \nabla \cdot \mathrm{P}_e + \mathrm{F}_{ei}$$

$$\mathrm{F}_{ei} = -m_e n_e (\mathbf{u}_e - \mathbf{u}_i)\nu_{ei} \qquad \mathrm{F}_{ie} = -m_i n_i (\mathbf{u}_i - \mathbf{u}_e)\nu_{ie}$$

- Conservation of momentum requires

$$\mathrm{F}_{ie} = -\mathrm{F}_{ei}$$

- $\boldsymbol{F}_{ie}$ ($\boldsymbol{F}_{ei}$) represent the momentum gain of the ion (electron) fluid due to the collisions with the electrons (ions)
- Since only Coulomb collisions are involved $\boldsymbol{F}_{ie}$ and $\boldsymbol{F}_{ei}$ will be proportional to the square of the charge (here considered as e^2)
- $\boldsymbol{F}_{ie}$ and $\boldsymbol{F}_{ei}$ must be also proportional to the ion and electron densities (here considered as n^2).
- Therefore, on physical grounds, it can be written

$$\mathrm{F}_{ei} = \eta e^2 n^2 (\mathbf{u}_i - \mathbf{u}_e)$$

- Comparing

$$\mathrm{F}_{ei} = \eta e^2 n^2 (\mathbf{u}_i - \mathbf{u}_e)$$

$$\mathrm{F}_{ei} = -m_e n_e (\mathbf{u}_e - \mathbf{u}_i) \nu_{ei}$$

it is readily found

$$\eta = \frac{m \nu_{ei}}{e^2 n^2}$$

- plasma specific resistivity

- By evaluating the electron-ion collision frequency (short-range, large angle collision) through a particle trajectory approximation it is found

$$\nu_{ei} = \frac{ne^4}{16\pi\epsilon^2{}_0 m^2 u^3}$$

(where u is the impact velocity) and therefore the resistivity is

$$\eta = \frac{m}{ne^2}\nu_{ei} = \frac{e^2}{16\pi\epsilon^2{}_0 m u^3}$$

- For a Maxwellian distribution

$$\frac{1}{2}mu^2 = \frac{1}{2}k_B T$$

$$\eta = \frac{\pi e^2 m^{1/2}}{(4\pi\varepsilon_0)^2 (k_B T_e)^{3/2}}$$

- The Spitzer resistivity includes a correction to provide better accuracy given by a factor *ln Λ*

(Coulomb logarithm) as

$$\eta = \frac{\pi e^2 m^{1/2}}{(4\pi\varepsilon_0)^2 (k_B T_e)^{3/2}} \ln \Lambda$$

The *Λ* represents the maximum impact parameter.

ln Λ is insensitive to the exact values of the plasma parameters. For most purposes, it will be sufficiently accurate to let it = 10 regardless of the type of plasma involved.

- In case a plasma current is carried only by the electrons, with $\boldsymbol{B}=0$ and with $T_e \sim 0$ (cold plasma) the electron fluid equation of motion in steady state is simply

$$0 = -en\mathbf{E} + \mathrm{F}_{ei}$$

- Since

$$\mathrm{F}_{ei} = \eta e^2 n^2 (\mathbf{u}_i - \mathbf{u}_e)$$

and

$$\mathbf{j} = en(\mathbf{u}_i - \mathbf{u}_e)$$

then

$$\mathbf{E} = \eta \mathbf{j}$$

that is the simplest form of the Ohm's law

- In a fully ionized plasma the plasma resistivity is independent on the density *n* because if the charge carriers increase also the collisional friction increase and the effects cancel out.

- In a weakly ionized plasma instead the collisional friction is due only to the neutrals and therefore does not increase with *n* and the current, for a given **E** is proportional to *n*.

- Fully ionized plasmas become "collisionless" at high temperature .