

Lecture -1

磁流体力学

Magnetohydrodynamics

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课程安排

课程	时间	内容
1	11.04	等离子体的流体近似
2	11.11	磁流体静力学
3	11.18	等离子体的冻结与磁重联
4	11.25	磁流体力学波
5	11.29	激波与间断面
6	12.02 (暂定)	磁流体动力学不稳定性
7	12.20 (暂定)	作业讲解与复习
8	12.27 (暂定)	期末考试

课程材料

空间天气学国家重点实验室网站
spaceweather.ac.cn



人才培养

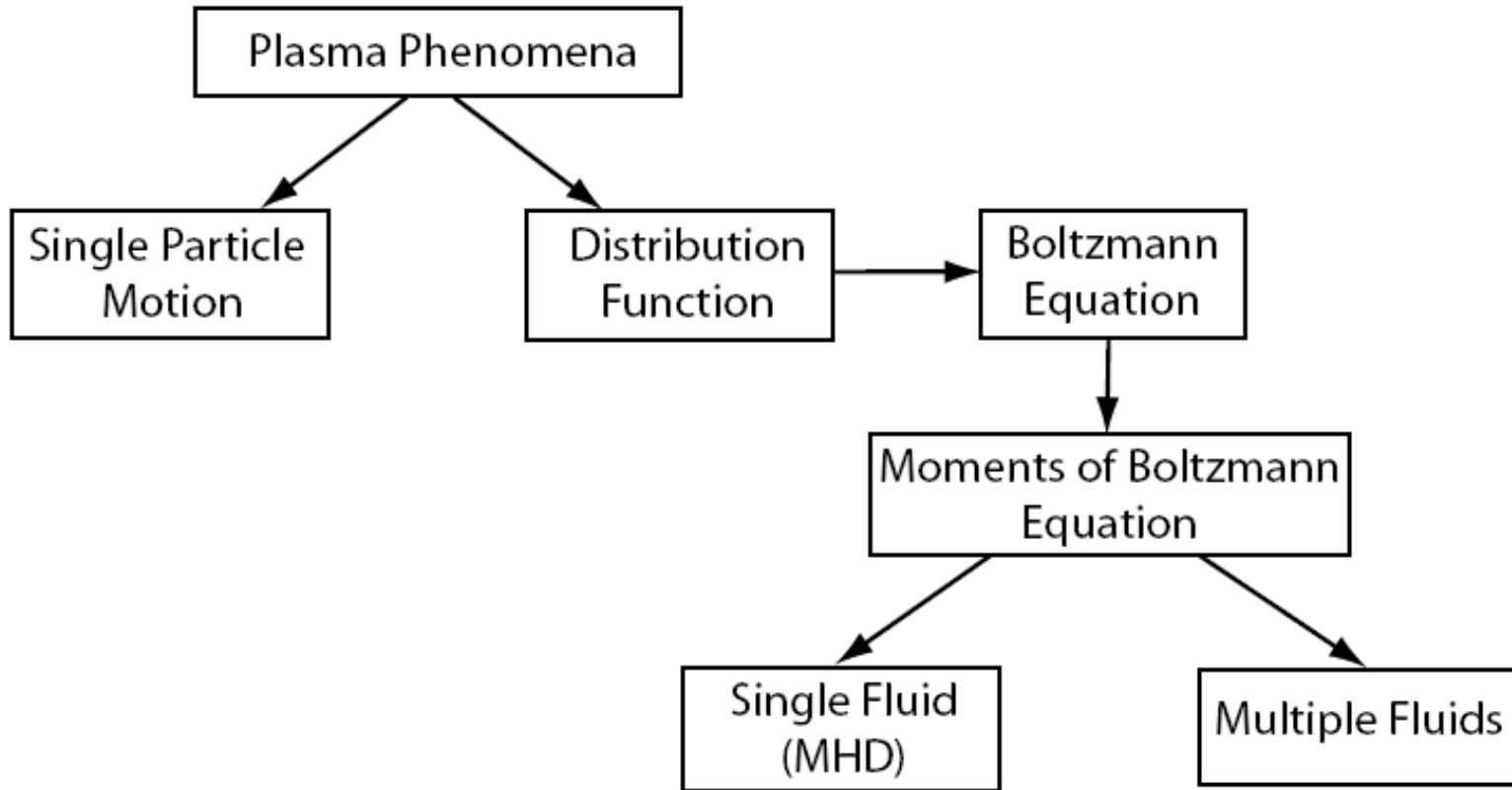


课程讲义

平时作业+考勤60%
期末考试40%



Hierarchy of Plasma Models



Fluid Approach

MHD Equations

Fluid Approach (1/2)

- ∅ The **E**, **B** fields are not prescribed but are determined by the positions and motions of the charges themselves.
- ∅ A typical plasma density might be 10^{12} ion-electron pairs per cm^3 . If each of these particles follows a complicated trajectory and it is necessary to follow each of these, predicting the plasma's behavior would be a hopeless task.
- ∅ Fortunately, this is not usually necessary, the majority of plasma phenomena observed in real experiments can be explained by a rather crude fluid model.

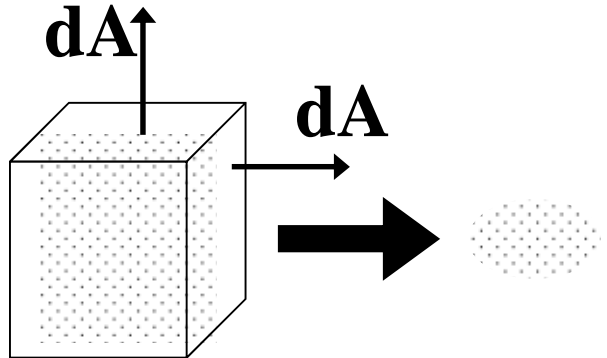
Fluid Approach

- ∅ This model is that used in fluid mechanics, in which the identity of the individual particle is neglected, and only the motion of fluid elements is taken into account. Of course, the fluid contains electrical charges.
- ∅ It is surprising that such a model works for plasmas, which generally have infrequent collisions. The ion and electron fluids will interact with each other even in the absence of collisions, because of the **E** and **B** fields are generated.

Fluid Eq. of Continuity

Conservation of matter requires that the number of particles (N) in a given volume (V) can only change if there is a net particle flux across the surface (S) bounding the Volume (V)

$$\frac{\partial N}{\partial t} = \frac{\partial}{\partial t} \int_V n dV = - \int_V \nabla \cdot (n\mathbf{u}) dV = - \int_V \nabla \cdot \tilde{\mathbf{N}} \cdot (n\mathbf{u}) dV$$

Divergence theorem


Particle flux

Since this must hold for any volume, it means

$$\frac{\partial n}{\partial t} + \tilde{\mathbf{N}} \cdot (n\mathbf{u}) = 0$$

Fluid Eq. of Motion

A. Neglecting collisions and thermal motion

$$m \frac{d\mathbf{v}}{dt} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

If neglect collisions and thermal effects, all particles in a fluid element move together with average velocity \mathbf{u}

$$nm \frac{d\mathbf{u}}{dt} = nq(\mathbf{E} + \mathbf{u} \times \mathbf{B})$$

d/dt is to be taken at the position of the fluid element, not very convenient. We wish to have equation for fluid elements fixed in space.



Transform to a fixed frame: (convective derivative)

$$\frac{dG}{dt} = \frac{\partial G}{\partial t} + (\mathbf{u} \cdot \tilde{\mathbf{N}})G$$

In a plasma with the fluid velocity \mathbf{u} , we can have:

$$nm \left[\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \tilde{\mathbf{N}})\mathbf{u} \right] = nq(\mathbf{E} + \mathbf{u} \times \mathbf{B})$$



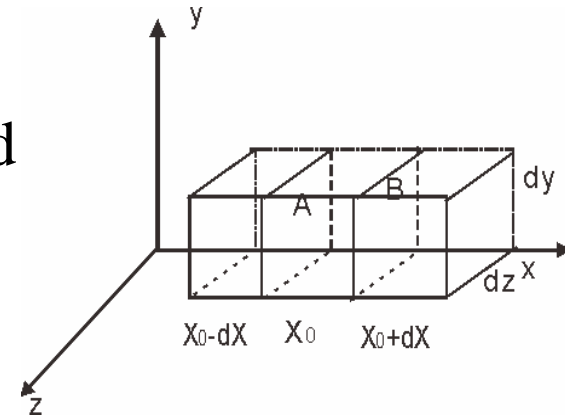
B. Including Thermal Effect (Pressure)

Thermal motion

=è the random motion of particles in and out of a fluid element

==è a pressure force should be added in the eq. of motion for a fluid element

(1) Consider only the x-component of motion through faces A and B of the fluid element centered at $(X_0, 1/2dy, 1/2dz)$.



The number of particles per volume with velocity V_x

$$Dn_v = Dv_x \iint f(v_x, v_y, v_z) dv_y dv_z$$

The number of particles per second passing through the face A with velocity V_x is

$$N = Dn_v v_x Dy Dz$$

Each particle carries a momentum. The momentum through face A, from particles with $V_x > 0$, in the fluid centered at $(x_0 - \Delta x, \frac{1}{2}\Delta y, \frac{1}{2}\Delta z)$

$$P_{A^+} = m \Delta y \Delta z \int_0^\infty v_x^2 f dv_x \int_{-\infty}^{\infty} dv_y \int_{-\infty}^{\infty} dv_z$$

$$n = \int_{-\infty}^{\infty} f dv_x \int_{-\infty}^{\infty} dv_y \int_{-\infty}^{\infty} dv_z \quad \langle v_x^2 \rangle = \frac{\int_0^\infty v_x^2 f dv_x \int_{-\infty}^{\infty} dv_y \int_{-\infty}^{\infty} dv_z}{\int_{-\infty}^{\infty} f dv_x \int_{-\infty}^{\infty} dv_y \int_{-\infty}^{\infty} dv_z}$$

(2) It can be written as:

$$P_{A^+} = m \Delta y \Delta z \frac{1}{2} n \langle v_x^2 \rangle \hat{u}_{x_0 - \Delta x}$$

The factor $\frac{1}{2}$ comes from the fact that only half the particles in the cube are going toward A.



Similarly, the momentum carried out through face B is

$$P_{B^+} = mDyDz \frac{1}{2} \frac{d}{dx} (n \langle v_x^2 \rangle) \dot{u}_{x_0}$$

(3) The net gain in momentum for particles:

$$\begin{aligned} P_{A^+} - P_{B^+} &= mDyDz \frac{1}{2} \frac{d}{dx} (n \langle v_x^2 \rangle)_{x_0 - Dx} - (n \langle v_x^2 \rangle)_{x_0} \dot{u} \\ &= \frac{1}{2} mDyDz (-Dx) \frac{d}{dx} (n \langle v_x^2 \rangle) \end{aligned}$$

This result is doubled by the contribution of left-moving particles, since they carry x momentum and also move in the opposite direction relative to the gradient. $n \langle v_x^2 \rangle$

The total change of momentum of the fluid element at X_0 is therefore

$$\frac{d}{dt} (nm u_x) Dx Dy Dz = -m \frac{d}{dx} (n \langle v_x^2 \rangle) Dx Dy Dz$$

(4) Let the velocity of a particle be composed by fluid velocity plus random thermal motion.

$$v_x = u_x + v_{xr}$$

For a 1D Maxwellian distribution, $\frac{1}{2} m \langle v_{xr}^2 \rangle = \frac{1}{2} kT$

$$\langle v_x^2 \rangle = u_x^2 + kT / m$$

then
$$\frac{\partial}{\partial t} (nm u_x) = - m \frac{\partial}{\partial x} [n(u_x^2 + kT / m)]$$

re-group
$$nm \left(\frac{\partial}{\partial t} u_x + u_x \frac{\partial}{\partial x} u_x \right) + m u_x \left(\frac{\partial}{\partial t} n + \frac{\partial}{\partial x} n u_x \right) = - \frac{\partial}{\partial x} (nkT)$$

Continuity
$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial x} n u_x = 0$$

$$nm \left(\frac{\partial}{\partial t} u_x + u_x \frac{\partial}{\partial x} u_x \right) = - \frac{\partial}{\partial x} p$$



Generalizing to three dimensions, we have

$$nm \left[\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} \right] = -\nabla p$$

Add the pressure-gradient force with the electromagnetic forces:

$$nm \left[\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} \right] = qn(\mathbf{E} + \mathbf{u} \times \mathbf{B}) - \nabla p$$



What we have derived is only a special case: the transfer of x momentum by motion in the x direction; and we have assumed that the fluid is isotropic, so that the same result holds in the y and z directions.

But it is possible to transfer y momentum by motion in the x direction. Suppose that the y -velocities of particles at $x_0 - \Delta x$ and $x_0 + \Delta x$ were a maximum, and that $v_y = 0$ at x_0 . Then particles passing through Faces A and B would bring more y -momentum into the fluid element at x_0 than they take out. This would give rise to a shear stress on the fluid element at x_0 , which must be described in general by a stress tensor, P . The off-diagonal elements of P are usually associated with viscosity.

The shear stress cannot be presented by a scalar p , but must be given by a tensor \mathbf{P} , whose components $P_{ij} = nm\overline{v_i v_j}$.

In the general case, the term ∇p is replaced by $\nabla \cdot \mathbf{P}$

$$nm\left[\frac{\nabla \mathbf{u}}{\nabla t} + (\mathbf{u} \cdot \tilde{\mathbf{N}})\mathbf{u}\right] = nq(\mathbf{E} + \mathbf{u} \times \mathbf{B}) - \tilde{\mathbf{N}} \cdot \mathbf{P}$$

When the distribution function is an isotropic

Maxwellian, $\mathbf{P} = \begin{pmatrix} p & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & p \end{pmatrix}$, $\nabla \cdot \mathbf{P}$ is just ∇p

C. Including Collisions

If a neutral gas is present, the charged fluid can exchange momentum with it through collisions. The momentum lost per collision will be proportional to the relative velocity between the charged fluid and the neutral gas ($\mathbf{u} - \mathbf{u}_0$), where \mathbf{u}_0 is the velocity of the neutral fluid. $\sigma = \frac{1}{\tau}$, where τ is the mean free time between collisions.

$$nm \left[\frac{d\mathbf{u}}{dt} + (\mathbf{u} \cdot \tilde{\mathbf{N}}) \mathbf{u} \right] = nq(\mathbf{E} + \mathbf{u} \times \mathbf{B}) - \tilde{\mathbf{N}} \cdot \mathbf{P} - \sigma mn(\mathbf{u} - \mathbf{u}_0)$$

- (A) When we took the velocity distribution to be Maxwellian, and therefore the **averages of velocity**, we assumed that there were collisions. However, the fluid theory is not sensitive to the distribution function, as long as we can use the same average velocity.
- (B) The other reason for the fluid theory to work for the plasma is that magnetic field and wave particle interaction may play the role as collisions in a certain sense.

Fluid Eq. of State

We use the thermodynamic equation of state to close the equations.

$$p = Cr^g$$

$$g = C_p / C_v$$

$$\implies \frac{\tilde{N}p}{p} = g \frac{\tilde{N}r}{r} = g \frac{\tilde{N}n}{n}$$

$$p_j n_j^{-g} = \text{const}$$

Fluid Eq. of State

(1) For isothermal compression: $\nabla p = kT\nabla n \Rightarrow \gamma = 1$

(2) Adiabatic compression (T also changes)

$$\frac{\nabla n}{n} + \frac{\nabla T}{T} = \gamma \frac{\nabla n}{n} \Rightarrow \frac{\nabla T}{T} = (\gamma - 1) \frac{\nabla n}{n}$$

(3) More general (adiabatic), $\gamma = (2+N)/N$

where N is the number of degrees of freedom, it is valid for negligible heat flow.

The validity of the equation of state requires that heat flow be negligible; that is, that thermal conductivity be low. Again, this more likely to be true in directions perpendicular to **B** than parallel to it.

Isotropic plasma and Anisotropic plasma

- ∅ A plasma is called isotropic, if its pressure tensor is diagonal with all diagonal elements having the same value $P = p I$
- ∅ Particle distribution in a plasma often anisotropic.

- Magnetized plasma: particle dynamics parallel to the field direction ($\vec{B} \parallel \vec{e}_z$) different from the perpendicular particle dynamics.

$$\Rightarrow \mathcal{P} = \begin{pmatrix} p_{\perp} & 0 & 0 \\ 0 & p_{\perp} & 0 \\ 0 & 0 & p_{\parallel} \end{pmatrix}.$$

- Both pressures related to a kinetic temperature by the ideal gas law

$$p_{\parallel} = nk_B T_{\parallel}$$

$$p_{\perp} = nk_B T_{\perp}$$

MultiFluid Equation:

$$r = n_e q_e + n_i q_i = e(-n_e + Zn_i)$$

$$\mathbf{j} = n_e q_e \mathbf{v}_e + n_i q_i \mathbf{v}_i = e(-n_e \mathbf{v}_e + Zn_i \mathbf{v}_i) = -en_e(\mathbf{v}_e - \mathbf{v}_i)$$

$$\frac{\partial n_j}{\partial t} + \tilde{\mathbf{N}} \cdot (n_j \mathbf{v}_j) = 0$$

$$n_j m_j \left[\frac{\partial \mathbf{v}_j}{\partial t} + (\mathbf{v}_j \cdot \tilde{\mathbf{N}}) \mathbf{v}_j \right] = n_j q_j (\mathbf{E} + \mathbf{v}_j \times \mathbf{B}) - \tilde{\mathbf{N}} p_j - \sum_{jk} \mathbf{s}_{jk} n_j m_j (\mathbf{v}_j - \mathbf{v}_k)$$

$$p_j n_j^{-g} = \text{const}$$

$$\tilde{\mathbf{N}} \cdot \mathbf{E} = \frac{r}{e_0} \quad \tilde{\mathbf{N}} \times \mathbf{B} = \mathbf{j} + m_0 e_0 \frac{\partial \mathbf{E}}{\partial t}$$

$$\tilde{\mathbf{N}} \times \mathbf{E} = - \frac{\partial \mathbf{B}}{\partial t} \quad \tilde{\mathbf{N}} \cdot \mathbf{B} = 0$$

Maxwell's Equations



Accounting
Unknowns

n_e, n	2
v_e, v_i	6
p_e, p_i	2
E, B	6
	<hr/>
	16

Equations

Continuity e,i	2
Momentum e,i	6
State e,i	2
Maxwell	8
	<hr/>
	18

But 2 of Maxwell ($\nabla \cdot$ equs) are redundant because can be deduced from others: e.g.

$$\nabla \cdot (\nabla \times \mathbf{E}) = 0 = \frac{\partial}{\partial t} (\nabla \cdot \mathbf{B})$$

and

$$\nabla \cdot (\nabla \times \mathbf{B}) = 0 = \mu_0 \nabla \cdot \mathbf{j} + \frac{1}{c^2} \frac{\partial (\nabla \cdot \mathbf{E})}{\partial t}$$

so 16 equs for 16 unknowns.

Magnetohydrodynamics (MHD)

∅ Description of plasma as a multi-fluid system is often too complicated.

Simplified to single fluid description (MHD)

∅ The MHD model is applicable only when charge separation (e.g. plasma oscillations or electromagnetic waves in plasmas) is negligible. The condition for it is that the length scales should be larger than the Debye length and the time scales larger than the inverse of plasma frequency.

Assuming that plasma consists of electrons (m_e , $q_e=-e$) and one component of ions (m_i , $q_i=e$)

Single-Fluid approximation

Mass density: $\rho_m = n_e m_e + n_i m_i$

Velocity: $\mathbf{V} = (n_e m_e \mathbf{v}_e + n_i m_i \mathbf{v}_i) / (n_e m_e + n_i m_i)$

Charge density: $\rho_q = n_e q_e + n_i q_i$

Current density: $\mathbf{j} = q_e n_e \mathbf{v}_e + q_i n_i \mathbf{v}_i = q_e n_e (\mathbf{v}_e - \mathbf{v}_i)$

Total pressure: $p = p_i + p_e$

Nsse

$$m_e(Ce) + m_i(Ci)$$

$$m_e \frac{\nabla n_e}{\nabla t} + m_e \tilde{\mathbf{N}} \cdot (n_e \mathbf{v}_e) = 0$$

$$m_i \frac{\nabla n_i}{\nabla t} + m_i \tilde{\mathbf{N}} \cdot (n_i \mathbf{v}_i) = 0$$

$$\begin{aligned} & \frac{\nabla (m_i n_i + m_e n_e)}{\nabla t} + \tilde{\mathbf{N}} \cdot (m_i n_i \mathbf{v}_i + m_e n_e \mathbf{v}_e) \\ &= \frac{\nabla (m_i n_i + m_e n_e)}{\nabla t} + \tilde{\mathbf{N}} \cdot \left[(m_i n_i + m_e n_e) \frac{m_i n_i \mathbf{v}_i + m_e n_e \mathbf{v}_e}{m_i n_i + m_e n_e} \right] \end{aligned}$$

Mass Conservation

$$\frac{\nabla r_m}{\nabla t} + \tilde{\mathbf{N}} \cdot (r_m \mathbf{V}) = 0$$

Nsse

$$q_e(Ce) + q_i(Ci)$$

$$q_e \frac{\nabla n_e}{\nabla t} + q_e \tilde{\mathbf{N}} \cdot (n_e \mathbf{v}_e) = 0$$

$$q_i \frac{\nabla n_i}{\nabla t} + q_i \tilde{\mathbf{N}} \cdot (n_i \mathbf{v}_i) = 0$$



Charge Conservation

$$\frac{\nabla q}{\nabla t} + \tilde{\mathbf{N}} \cdot \mathbf{j} = 0$$

Nsse
(Me) + (Mi)

$$n_j m_j \left[\frac{\mathbf{v}_j}{t} + (\mathbf{v}_j \cdot \tilde{\mathbf{N}}) \mathbf{v}_j \right] = n_j q_j (\mathbf{E} + \mathbf{v}_j \times \mathbf{B}) - \tilde{\mathbf{N}} p_j - \mathbf{s}_{jk} n_j m_j (\mathbf{v}_j - \mathbf{v}_k)$$

RHS:====è

$$\mathring{\mathbf{a}}_j \{ n_j q_j (\mathbf{E} + \mathbf{v}_j \times \mathbf{B}) - \tilde{\mathbf{N}} p_j + \mathbf{F}_{jk} \} = r_q \mathbf{E} + \mathbf{j} \times \mathbf{B} - \tilde{\mathbf{N}} (p_e + p_i)$$

($\mathbf{F}_{ei} = -\mathbf{F}_{ie}$, so no net friction)

LHS==è : $\mathring{\mathbf{a}}_j m_j n_j \left[\frac{\mathbf{v}_j}{t} + \mathbf{v}_j \cdot \tilde{\mathbf{N}} \right] \mathbf{v}_j$

(1) ignore the contribution of electron momentum ($m_e \ll m_i$)

(2) V is approximated as \mathbf{v}_i

$$\mathring{\mathbf{a}}_j m_j n_j \left[\frac{\mathbf{v}_j}{t} + \mathbf{v}_j \cdot \tilde{\mathbf{N}} \right] \mathbf{v}_j \gg r_m \left(\frac{\mathbf{v}_j}{t} + \mathbf{V} \cdot \tilde{\mathbf{N}} \right) \mathbf{V}$$

Momentum Eq.

$$r_m \left[\frac{\mathbf{v}_j}{t} + \mathbf{V} \cdot \tilde{\mathbf{N}} \right] \mathbf{V} = r_q \mathbf{E} + \mathbf{j} \times \mathbf{B} - \tilde{\mathbf{N}} p$$

Nsse

$$\frac{q_e}{m_e} (Me) + \frac{q_i}{m_i} (Mi)$$

$$n_j m_j \left[\frac{d\mathbf{v}_j}{dt} + (\mathbf{v}_j \cdot \tilde{\mathbf{N}}) \mathbf{v}_j \right] = n_j q_j (\mathbf{E} + \mathbf{v}_j \times \mathbf{B}) - \tilde{\mathbf{N}} p_j - \mathbf{S}_{jk} n_j m_j (\mathbf{v}_j - \mathbf{v}_k)$$

$$\dot{\mathbf{a}}_j n_j q_j \left[\frac{d\mathbf{v}_j}{dt} + \mathbf{v}_j \cdot \tilde{\mathbf{N}} \right] \mathbf{v}_j = \dot{\mathbf{a}}_j \left[\frac{n_j q_j^2}{m_j} (\mathbf{E} + \mathbf{v}_j \times \mathbf{B}) - \frac{q_j}{m_j} \tilde{\mathbf{N}} p_j + \frac{q_j}{m_j} \mathbf{F}_{jk} \right]$$

Nsse

$$\mathbf{r}_m = n_e m_e + n_i m_i \quad \mathbf{j} = n_e q_e \mathbf{v}_e + n_i q_i \mathbf{v}_i = n_e q_e (\mathbf{v}_e - \mathbf{v}_i)$$

$$\mathring{\mathbf{a}}_j n_j q_j \left[\frac{\mathbf{1}}{\|\mathbf{t}\|} + \mathbf{v}_j \cdot \tilde{\mathbf{N}} \right] \mathbf{v}_j \gg \mathring{\mathbf{a}}_j n_j q_j \left[\frac{\mathbf{1}}{\|\mathbf{t}\|} \mathbf{v}_j \right] = n_i q_i \frac{\|\mathbf{v}_i\|}{\|\mathbf{t}\|} + n_e q_e \frac{\|\mathbf{v}_e\|}{\|\mathbf{t}\|}$$

$$\begin{aligned} \mathbf{r}_m \frac{\|\mathbf{j}\|}{\|\mathbf{t}\|} \left(\frac{\mathbf{j}}{\mathbf{r}_m} \right) &= (n_e m_e + n_i m_i) \frac{\|\mathbf{j}\|}{\|\mathbf{t}\|} \left(\frac{n_e q_e \mathbf{v}_e + n_i q_i \mathbf{v}_i}{n_e m_e + n_i m_i} \right) \\ &= \left(n_i + \frac{n_e m_e}{m_i} \right) \frac{\|\mathbf{j}\|}{\|\mathbf{t}\|} \frac{n_e q_e \mathbf{v}_e + n_i q_i \mathbf{v}_i}{n_i + \frac{n_e m_e}{m_i}} \gg n_i \frac{\|\mathbf{j}\|}{\|\mathbf{t}\|} \frac{n_e q_e \mathbf{v}_e + n_i q_i \mathbf{v}_i}{n_i} \end{aligned}$$

$$\text{LHS} \Rightarrow \mathring{\mathbf{a}}_j n_j q_j \left[\frac{\mathbf{1}}{\|\mathbf{t}\|} + \mathbf{v}_j \cdot \tilde{\mathbf{N}} \right] \mathbf{v}_j \gg \mathbf{r}_m \frac{\|\mathbf{j}\|}{\|\mathbf{t}\|} \left(\frac{\mathbf{j}}{\mathbf{r}_m} \right)$$

Nsse

RHS:====è

[[1]]

$$\begin{aligned} \sum_j \frac{n_j q_j^2}{m_j} \mathbf{E} &= n_e^2 q_e^2 \left(\frac{1}{n_e m_e} + \frac{1}{n_i m_i} \right) \mathbf{E} \\ &= n_e^2 q_e^2 \left(\frac{n_i m_i + n_e m_e}{n_e m_e n_i m_i} \right) \mathbf{E} \\ &= - \frac{q_e q_i}{m_e m_i} r_m \mathbf{E} \end{aligned}$$

$$\mathring{\mathbf{a}}_j \frac{n_j q_j^2}{m_j} \mathbf{v}_j = n_e q_e^2 \frac{\mathbf{v}_e}{m_e} + n_i q_i^2 \frac{\mathbf{v}_i}{m_i}$$

$$= \frac{q_e q_i}{m_e m_i} \left\{ \frac{n_e q_e m_i}{q_i} \mathbf{v}_e + \frac{n_i q_i m_e}{q_e} \mathbf{v}_i \right\}$$

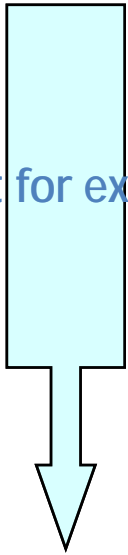
$$= - \frac{q_e q_i}{m_e m_i} \{ n_i m_i \mathbf{v}_e + n_e m_e \mathbf{v}_i \}$$

$$= - \frac{q_e q_i}{m_e m_i} \left\{ n_e m_e \mathbf{v}_e + n_i m_i \mathbf{v}_i - \left(\frac{m_i}{q_i} + \frac{m_e}{q_e} \right) (q_e n_e \mathbf{v}_e + q_i n_i \mathbf{v}_i) \right\}$$

$$= - \frac{q_e q_i}{m_e m_i} \left\{ \mathbf{r}_m \mathbf{V} - \left(\frac{m_i}{q_i} + \frac{m_e}{q_e} \right) \mathbf{j} \right\}$$

$[n_i q_i = -n_e q_e]$

Left for excise



Nsse

[III]

$$\mathbf{F}_{ei} = -\mathbf{s}_{ei} n_e m_e (\mathbf{v}_e - \mathbf{v}_i) = -\mathbf{F}_{ie}$$

$$\mathring{\mathbf{a}}_j \frac{q_j}{m_j} \mathbf{F}_{jk} = -\mathbf{s}_{ei} \left(n_e q_e - n_e q_i \frac{m_e}{m_i} \right) (\mathbf{v}_e - \mathbf{v}_i)$$

$$= -\mathbf{s}_{ei} \left(1 - \frac{q_e m_e}{q_i m_i} \right) \mathbf{j}$$



LHS=RHS

$$\mathbf{r}_m \frac{\nabla}{\nabla t} \left(\frac{\mathbf{j}}{\mathbf{r}_m} \right) = \frac{q_e q_i}{m_e m_i} \left[\mathbf{r}_m \mathbf{E} + \left\{ \mathbf{r}_m \mathbf{V} - \left(\frac{m_i}{q_i} + \frac{m_e}{q_e} \right) \mathbf{j} \right\}' \mathbf{B} \right. \\ \left. - \left(\frac{q_e}{m_e} \tilde{N} p_e + \frac{q_i}{m_i} \tilde{N} p_i \right) - \left(1 - \frac{q_e m_e}{q_i m_i} \right) \mathbf{s}_{ei} \mathbf{j} \right]$$

$$\mathbf{E} + \mathbf{V}' \mathbf{B} = - \frac{m_e m_i}{q_e q_i} \frac{\nabla}{\nabla t} \left(\frac{\mathbf{j}}{\mathbf{r}_m} \right) + \frac{1}{\mathbf{r}_m} \left(\frac{m_i}{q_i} + \frac{m_e}{q_e} \right) \mathbf{j}' \mathbf{B} \\ - \left(\frac{q_e}{m_e} \tilde{N} p_e + \frac{q_i}{m_i} \tilde{N} p_i \right) \frac{m_e m_i}{q_e q_i \mathbf{r}_m} - \left(1 - \frac{q_e m_e}{q_i m_i} \right) \frac{m_e m_i}{q_e q_i \mathbf{r}_m} \mathbf{s}_{ei} \mathbf{j}$$

General Ohm's Law

$$\mathbf{E} + \mathbf{V}' \times \mathbf{B} = - \frac{m_e m_i}{q_e q_i} \frac{\partial}{\partial t} \left(\frac{\mathbf{j}}{r_m} \right) + \frac{1}{r_m} \left(\frac{m_i}{q_i} + \frac{m_e}{q_e} \right) \mathbf{j}' \times \mathbf{B}$$

$$- \left(\frac{q_e \tilde{N} p_e}{m_e} + \frac{q_i \tilde{N} p_i}{m_i} \right) \frac{m_e m_i}{q_e q_i r_m}$$

$$- \left(1 - \frac{q_e m_e}{q_i m_i} \right) \frac{m_e m_i}{q_e q_i r_m} \mathbf{s}_{ei} \times \mathbf{j}$$

Hall term

electron inertia,
negligible for low
enough frequency

~ the resistivity

$$\mathbf{E} + \mathbf{V}' \times \mathbf{B} = \eta \mathbf{j} + \frac{1}{ne} \mathbf{j}' \times \mathbf{B} - \frac{1}{ne} \tilde{\mathbf{N}} \cdot \mathbf{P}_e + \frac{m_e}{ne^2} \frac{\partial \mathbf{j}}{\partial t}$$

General Ohm's Law

Last term in \mathbf{J} has a coefficient, ignoring m_e/m_i c.f. 1

$$\frac{m_e m_i \overline{v_{ei}}}{q_e q_i (n_i m_i)} = \frac{m_e \overline{v_{ei}}}{q_e^2 n_e} = \eta \quad (\sigma_{ei} = \overline{v_{ei}})$$

The resistivity

Hence dropping electron inertia, Hall term, pressure term, the Ohm's law becomes:

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = \eta \mathbf{j}$$

Equation for the State

$$p_e n_e^{-\gamma_e} + p_i n_i^{-\gamma_i} = \text{const.}$$

$$g_e = g_i = g$$

$$r_m \gg nm_i$$

$$p r_m^{-g} = \text{const}$$

Summary of MHD equations (I)

Mass Conservation $\frac{\partial \rho_m}{\partial t} + \tilde{\mathbf{N}} \cdot (\rho_m \mathbf{V}) = 0$

Momentum $\rho_m \left(\frac{\partial \mathbf{V}}{\partial t} + \mathbf{V} \cdot \tilde{\mathbf{N}} \right) \mathbf{V} = \rho_q \mathbf{E} + \mathbf{j}' \times \mathbf{B} - \tilde{\mathbf{N}} p$

Eq. State $\rho_m^{-\gamma} = \text{const}$

Charge Conservation $\frac{\partial \rho_q}{\partial t} + \tilde{\mathbf{N}} \cdot \mathbf{j} = 0$

Ohm's Law $\mathbf{E} + \mathbf{v} \times \mathbf{B} = \eta \mathbf{j}$

- Since the fluid is assumed to be electrically neutral, the charge density ρ_q is taken to be zero.
- We shall not need Poisson's equation because that is taken care of by quasi-neutrality
- Normally, one uses MHD only for low frequency phenomena, so the Maxwell displacement current $\frac{1}{c^2} \frac{\partial E}{\partial t}$ can be ignored in comparison with the conduction current.

Summary of MHD Equations (II)

In order to obtain a complete and self-consistent description of a resistive MHD plasma, the fluid equations and Ohm's law must be combined with Maxwell's equations:

Ampere's Law $\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$

$$\tilde{\mathbf{N}} \cdot \mathbf{B} = 0$$

Faraday's law $\tilde{\mathbf{N}}' \cdot \mathbf{E} = - \frac{\nabla \cdot \mathbf{B}}{t}$

Gauss's law $\nabla \cdot \mathbf{E} = 0$

Ideal MHD equations

As the collision frequency goes to zero, the conductivity goes to infinity: ideal MHD plasma

$$\mathbf{E} + \mathbf{V} \times \mathbf{B} = 0$$

The fluid velocity component perpendicular to \mathbf{B} is then given by $\mathbf{E} \times \mathbf{B} / B^2$, which is identical to the $\mathbf{E} \times \mathbf{B}$ drift velocity encountered in single particle orbit theory.