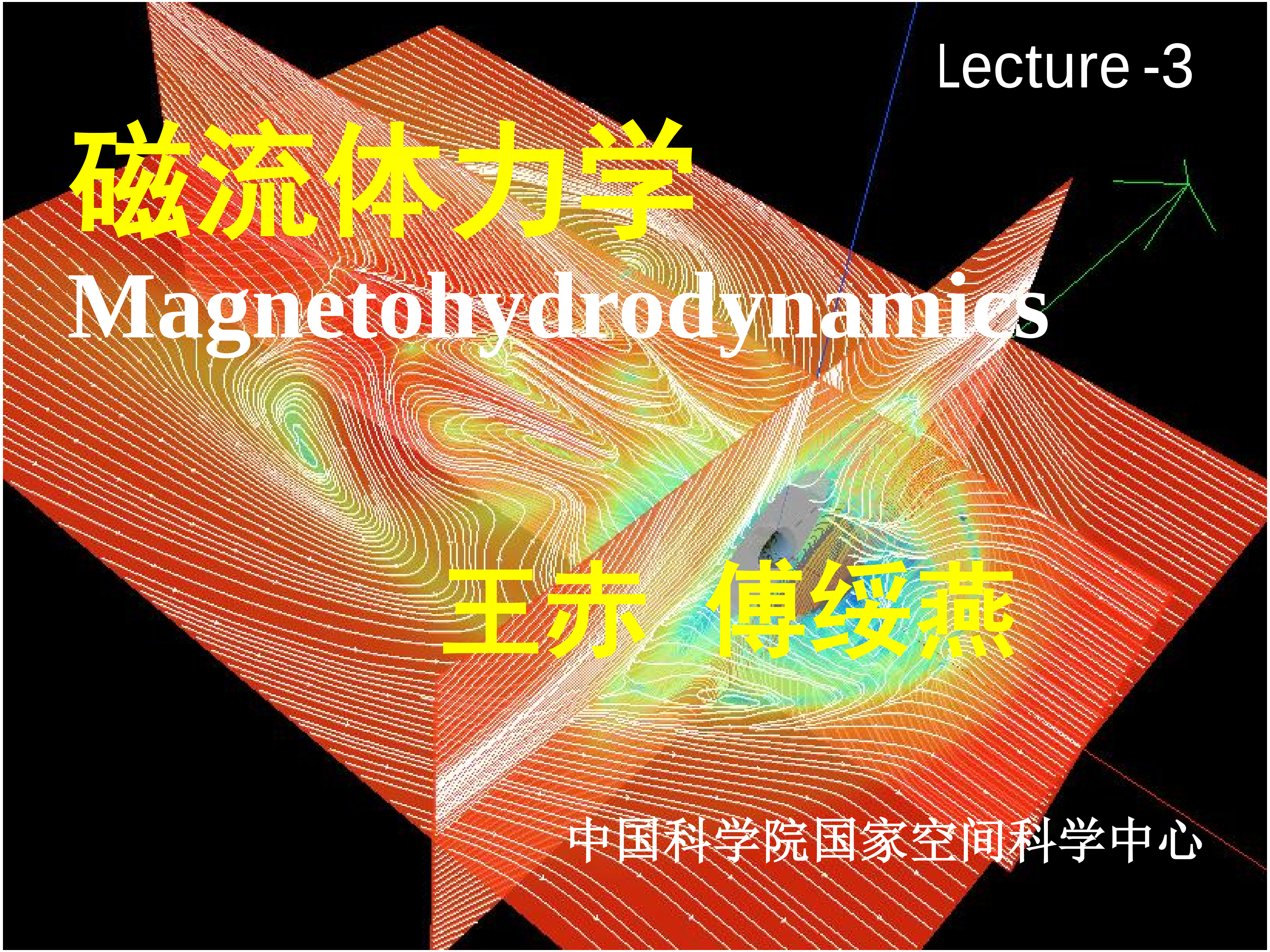
Lecture -3

磁流体力学

Magnetohydrodynamics



王赤 傅绥燕

中国科学院国家空间科学中心

Magnetic field convection, diffusion, reconnection

- Magnetic diffusion
- Frozen field theorem
- Magnetic reconnection

Convection and Diffusion

Starting from the Ohm's Law $$\mathbf{E}+\mathbf{V}\times\mathbf{B}=\frac{\mathbf{J}}{\sigma}$$

$$\Longrightarrow \quad \nabla\times\mathbf{E}=-\nabla\times\left(\mathbf{V}\times\mathbf{B}-\frac{\mathbf{J}}{\sigma}\right)$$

$$\Longrightarrow \quad \frac{\partial\mathbf{B}}{\partial t}=\nabla\times\left(\mathbf{V}\times\mathbf{B}-\frac{\mathbf{J}}{\sigma}\right)$$

$$\mathbf{J}=\frac{1}{\mu_0}\nabla\times\mathbf{B} \qquad \nabla\times(\nabla\times\boldsymbol{B})=\nabla(\nabla\cdot\boldsymbol{B})-\nabla^2 B$$

$$\Longrightarrow \quad \frac{\partial\mathbf{B}}{\partial t}=\nabla\times(\mathbf{V}\times\mathbf{B})+\frac{1}{\mu_0\sigma}\nabla^2\mathbf{B}$$

convection term diffusion term

Magntic Reynolds Number

To determine which term dominates, it is useful to introduce a dimensionless parameter $\boldsymbol{R_{m}}$, Which is defined as the ratio of the typical magnitude of the convection term to the typical magnitude of the diffusion term.

$$R_m = \frac{|\nabla \times (\boldsymbol{V} \times \boldsymbol{B})|}{\left|\frac{1}{\mu_0 \sigma} \nabla^2 \boldsymbol{B}\right|}$$

To make a rough estimate of the R_m, replace ∇ by 1/L, where L is a length scale that characterizes the spatial gradients in the fluid, and ignore the vector character of the equation.

$$R_m = \mu_0 \sigma L V = LV/D_m$$

If $R_m \gg 1$, the convection term dominates, and if $R_m \ll 1$, the diffusion term dominates. For example, Rm ~ 10^9 for sunspot

Diffusion

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{V} \times \mathbf{B}) + \frac{1}{\mu_0 \sigma} \nabla^2 \mathbf{B}$$

if $R_m \ll 1$, the diffusion term dominates

$$\frac{\partial \mathbf{B}}{\partial t} = D_m \nabla^2 \mathbf{B}$$

$$D_m = \frac{1}{\mu_0 \sigma}$$ *Magnetic viscosity*

Dimensional analysis: $$\frac{B}{\tau_d} = \frac{D_m}{L^2} B$$

Diffusion time scale is: $$\tau_d = \mu_0 \sigma L^2$$

For most geophysical and astrophysical applications, the diffusion times are extremely long, sometimes even longer than the age of the Universe. For solar atmosphere, τ_d~10 billion year

Energy Convension

Magnetic diffusion corresponds to electric current dissipation, which converts the magnetic energy into the thermal and kinetic energy of the plasma.

Total magnetic energy: $W_m = \frac{1}{2\mu_0}\int \boldsymbol{B}^2 \mathrm{dV}$

$$\frac{\partial W_m}{\partial t} = \frac{1}{\mu_0}\int \boldsymbol{B}\cdot\frac{\partial \boldsymbol{B}}{\partial t}\ \mathrm{dV}$$

$$\boldsymbol{B}\cdot\frac{\partial \boldsymbol{B}}{\partial t} = \boldsymbol{B}\cdot\frac{1}{\mu_0\sigma}\nabla^2\boldsymbol{B} = -\boldsymbol{B}\cdot\frac{\eta}{\mu_0}\nabla\times(\nabla\times\boldsymbol{B}) = -\boldsymbol{B}\cdot\eta\nabla\times\boldsymbol{J}$$

$$\boldsymbol{B}\cdot(\boldsymbol{\nabla}\times\mathbf{J}) = \boldsymbol{\nabla}\cdot(\boldsymbol{J}\times\boldsymbol{B}) + \boldsymbol{J}\cdot(\boldsymbol{\nabla}\times\boldsymbol{B})$$

Therefore (The volume integral of the first term can be evaluated by the surface Integral at infinite where both B and J are zero.

$$\frac{\partial W_m}{\partial t} = \frac{1}{\mu_0}\int \boldsymbol{B}\cdot\frac{\partial \boldsymbol{B}}{\partial t}\ \mathrm{dV} = -\int \eta J^2$$

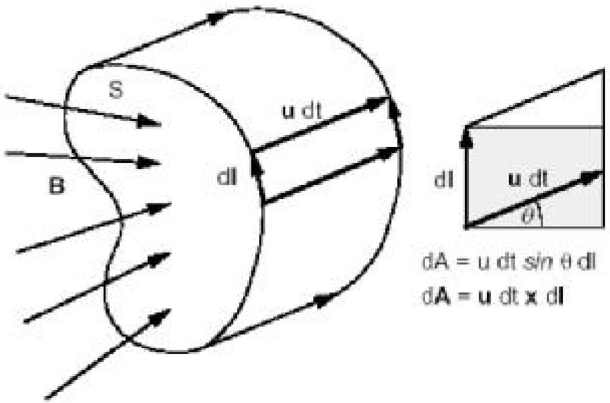
Joule heating

Frozen Field Theorem

Large magnetic Reynolds numbers occur whenever the conductivity, the fluid velocity, and the length scale are sufficiently large to make $R_m \gg 1$.

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B})$$

This relation will allow us to study the behavior of magnetic flux through a surface moving with the plasma at velocity u.



Frozen Field Theorem

The magnetic flux through the surface S is

$$\Phi_s = \int \mathbf{B} \cdot d\mathbf{s}$$

The flux changes either through time variations in B or due to the surface moving with the plasma fluid element to a new position where B is different – the convective term. The first contribution is just.

$$\frac{\partial \Phi_s}{\partial t} = \int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{s}$$

Frozen Field Theorem

To evaluate the convective component, let *dl* be a unit vector lying on the circumference of S. Due to the motion of the plasma fluid element, this element sweeps out an area $d\mathbf{A} = \mathbf{u}dt \times d\mathbf{l}$ in time *dt.* The associated change in magnetic flux through this area element is $d\Phi = \mathbf{B} \cdot d\mathbf{A}$

The total rate of change of flux is found by integrating along around the circumference of S:

$$\frac{d\Phi}{dt} = \oint \mathbf{B} \cdot (\mathbf{u} \times d\mathbf{l})$$

$$= \oint (\mathbf{B} \times \mathbf{u}) \cdot d\mathbf{l}$$

$$= \int \nabla \times (\mathbf{B} \times \mathbf{u}) \cdot d\mathbf{S}$$

← Stokes' theorem

Frozen Field Theorem

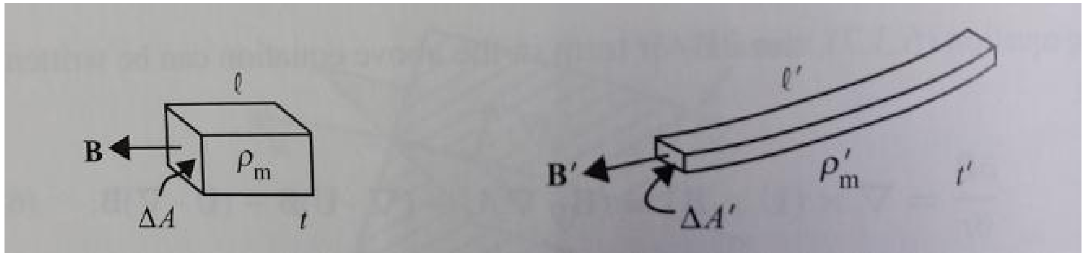
If now we add the two components of the changing flux, we obtain

$$\frac{D\Phi_s}{Dt} = \int [\frac{\partial \mathbf{B}}{\partial t} - \nabla \times (\mathbf{B} \times \mathbf{u})] \cdot d\mathbf{S}$$
$$= 0$$

The magnetic flux through S does not change with time. This implies that the magnetic lines of force are “frozen” into the plasma and are transported with it. Thus, if the plasma expands, the field strength decreases (reminiscent of magnetic moment). If the flux did change, there would be an induced emf established to oppose the change (Lenz’s law). This would establish and E x B drift that would change plasma shape to preserve the magnetic flux.

Example: Magnetic field amplified

An interesting consequence of the frozen field theorem is that the magnetic field strength can be amplified by changes in the fluid geometry or the mass density.

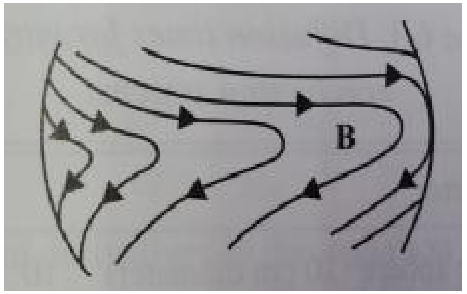


The frozen field theorem shows $B\Delta A = B'\Delta A'$, from mass conservation, one also has the relation $\rho_m \Delta A l = \rho'_m \Delta A' l'$. Eliminating ΔA, which gives

$$B' = B\left(\frac{\rho'_m}{\rho_m}\right)\left(\frac{l'}{l}\right)$$

Example: Magnetic field amplified

An example of magnetic field amplification occurs when a star collapses to form a neutron star. As the star collapses, the density increases much fast than the length decreases, so the magnetic field is increased by a factor of $(l/l')^2$, which can be very large.



An example of magnetic amplification due to stretching occurs in the solar atmosphere. Because the Sun rotates faster near the equator than near the poles, magnetic lines near the equator are stretched azimuthally around the Sun. Since the stretching motion is nearly horizontal, the density remains essentially constant. As the length of the field line increases, the magnetic field strength increases. Eventually the magnetic field becomes so strong that the Rayleigh-Taylor instability causes the magnetic field lies to bugle up through the surface of the Sun.

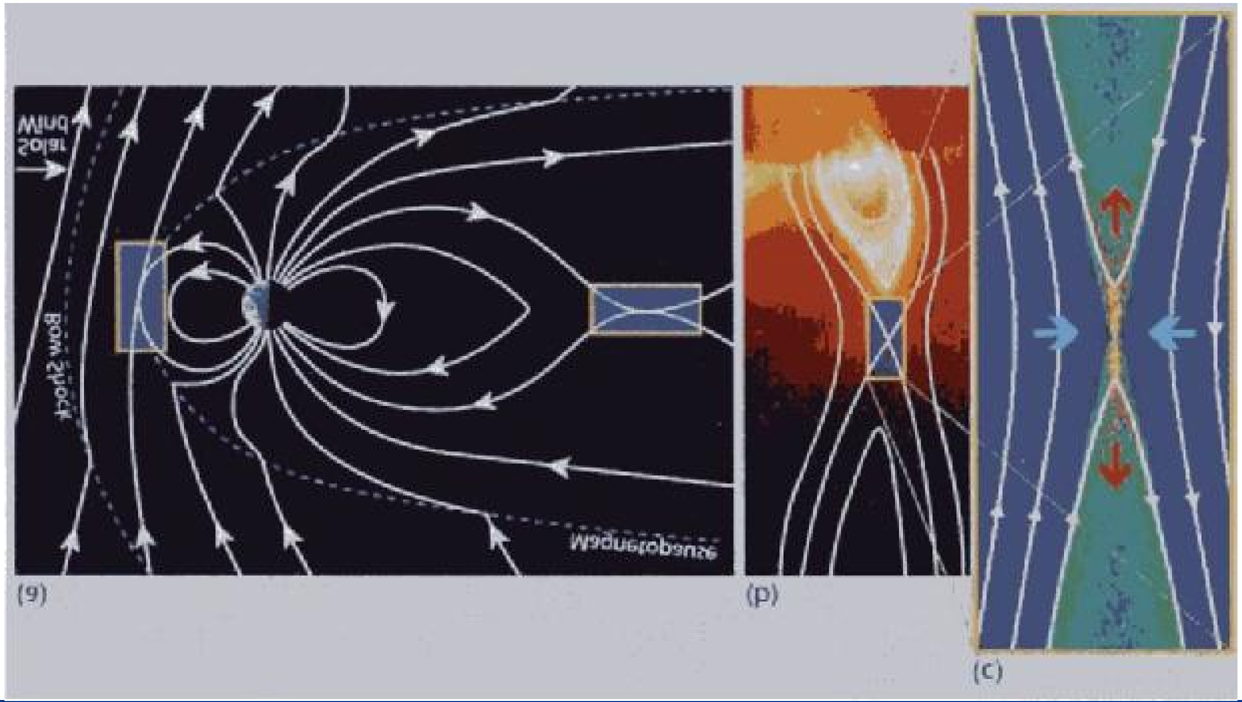
Magnetic Reconnection

For most geophysical and astrophysical applications, the magnetic field can be considered to be “frozen” into the fluid (i.e. Rm >> 1). For example, Rm ~ 10^{11} for magnetosphere, implying that the frozen-in-flux condition holds to a very high accuracy.

However, even in such large-scale systems, the frozen field concept sometime breaks down in small localized regions such as the thin boundaries between different plasma regimes where the scale size becomes very small, and so MHD can and does break down locally.

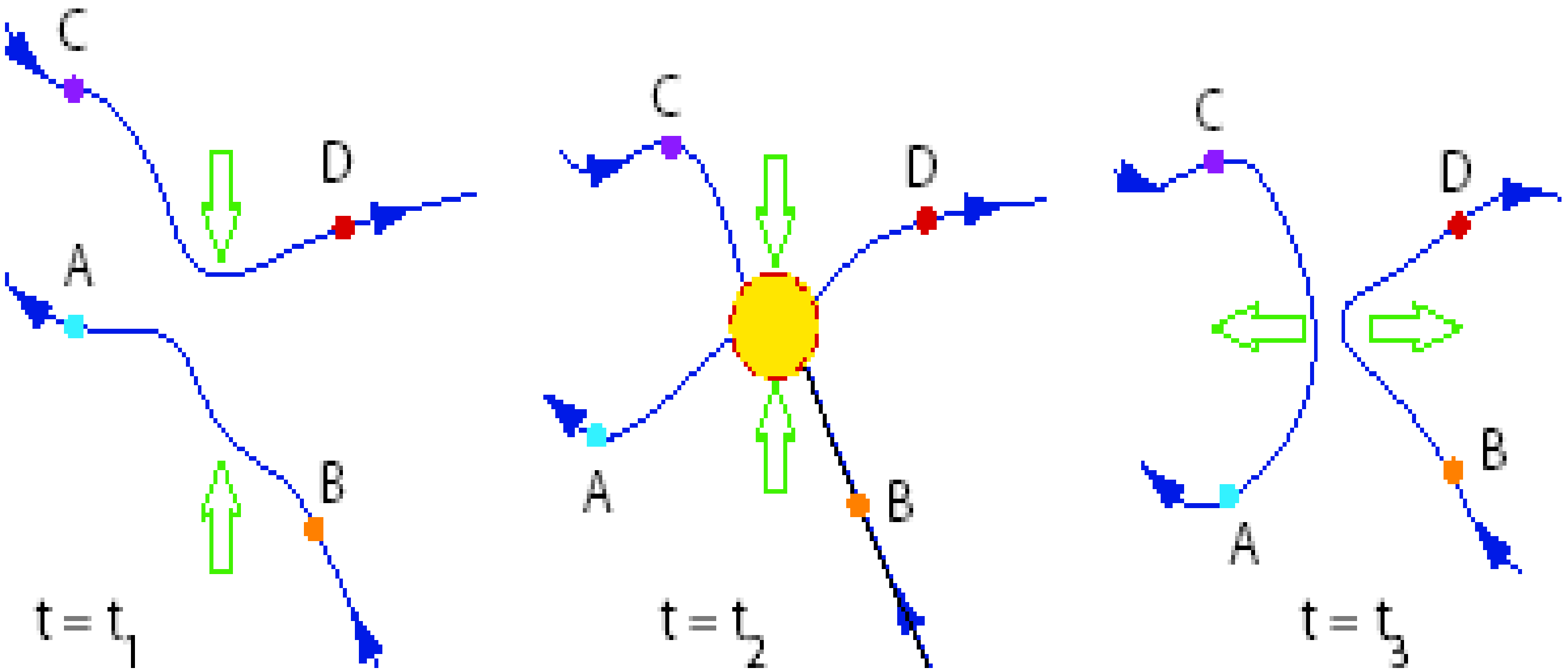
Magnetic Reconnection

Magnetic reconnection is the fundamental mechanism by which magnetic energy is dissipated in the universe. Observationally, energy is released in bursts rather than in a continuous manner, driving phenomena such as solar flares and magnetospheric substorms.



The Meaning of Reconnection

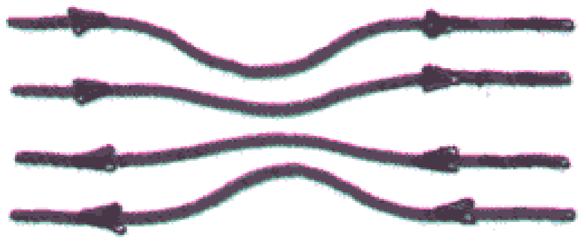
"a process where plasma flows across a surface that separates regions containing topologically different field lines" [Vasyliunas, 1975]



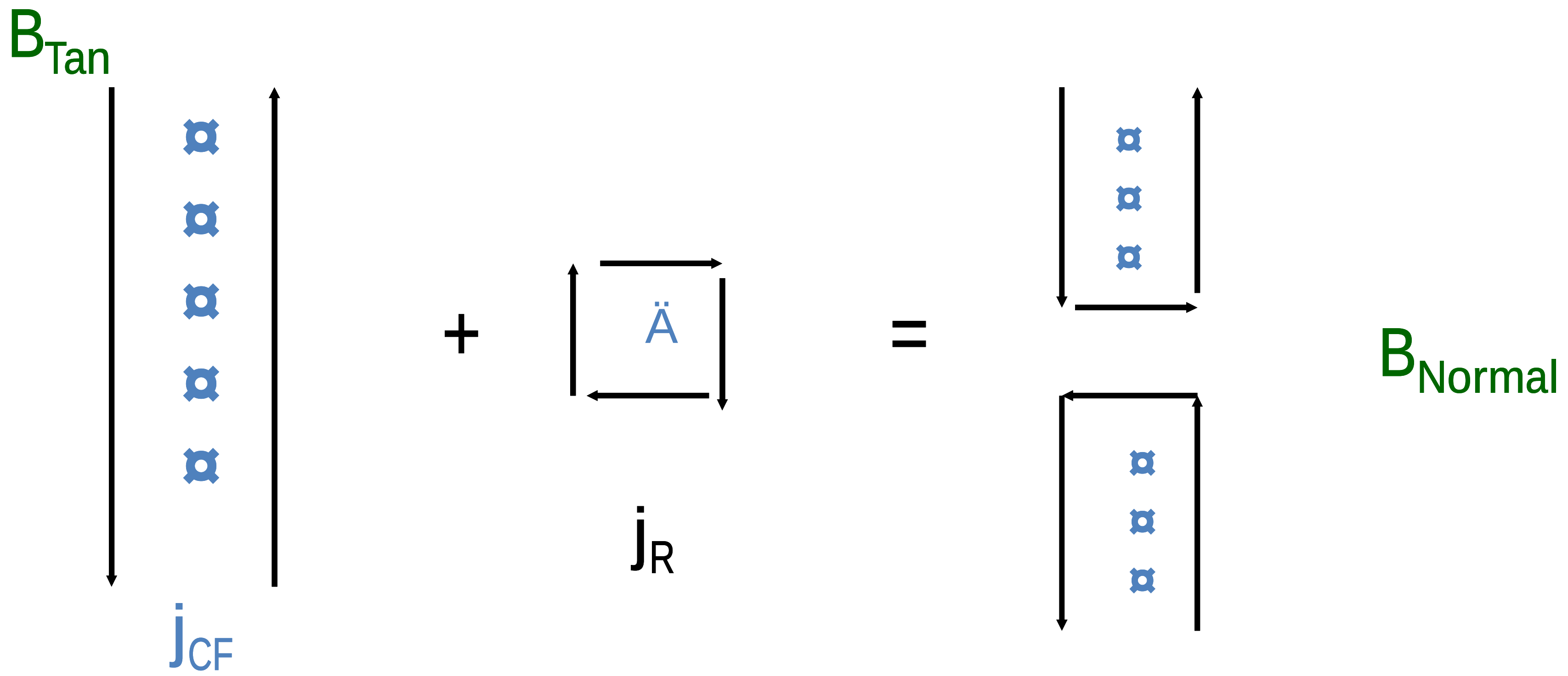
Axford 1984

Role of Resistivity

- The frozen-in condition implies that in an ideal plasma (h=0) no topological change in the magnetic field is possible
  - tubes of magnetic flux are preserved



  - magnetic reconnection requires resistivity or some other dissipation mechanism



The counter current j_R leads to a reconfiguration of the fieldlines

Diffusion of B

Consider first a 1-D current sheet with oppositely directed magnetic fields
Assume $\mathbf{V}= 0$. The induction equation reduces to diffusion equation:

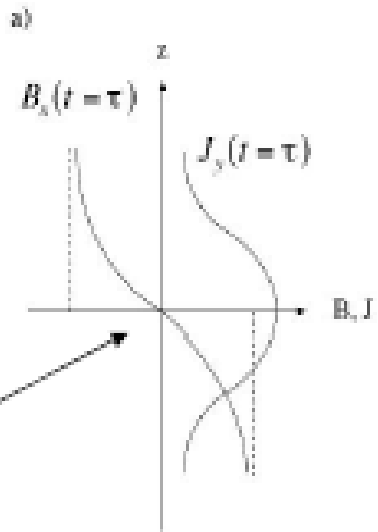
$$\frac{\partial B_x}{\partial t} = \frac{1}{\mu_0 \sigma} \frac{\partial^2 B_x}{\partial z^2}$$

Let the current sheet be infinitely thin at $t = 0$

The field diffuses as

$$B_x(z) = B_0 \mathrm{erf}\left\{ \left(\frac{\mu_0 \sigma}{4t} \right)^{1/2} z \right\}$$

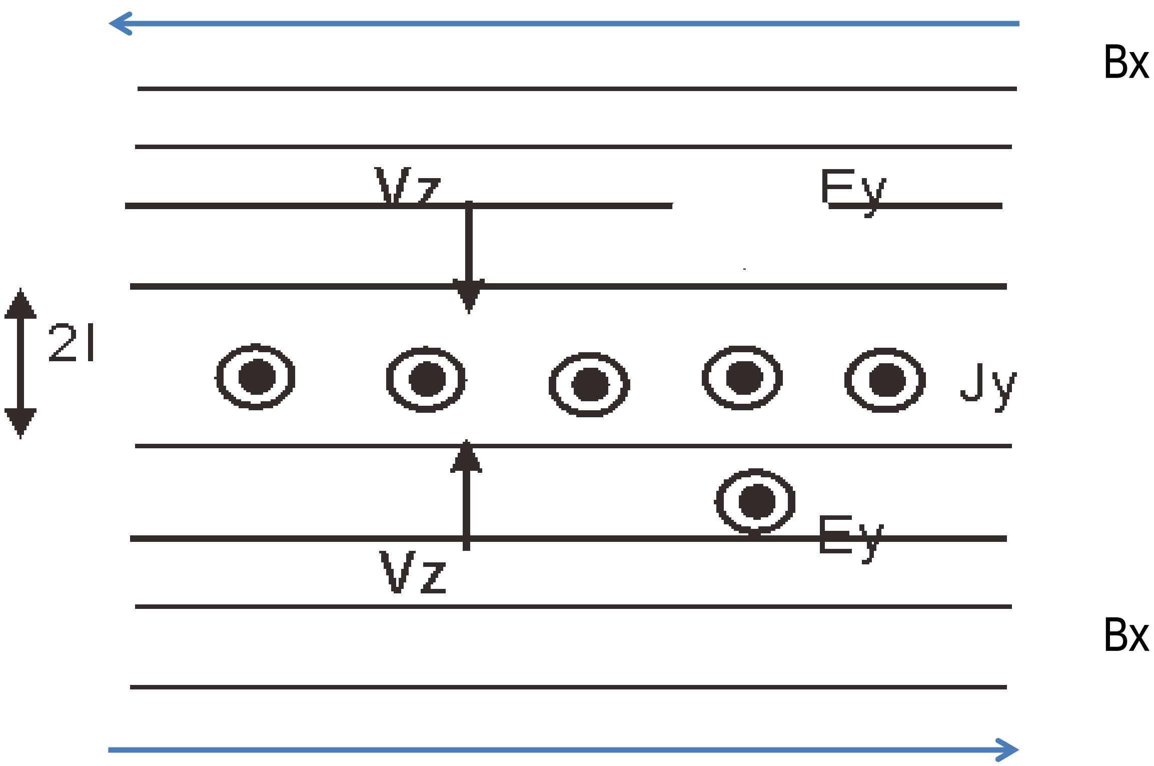
$$\mathrm{erf}(u) = \frac{2}{\sqrt{\pi}} \int e^{-v^2} dv$$



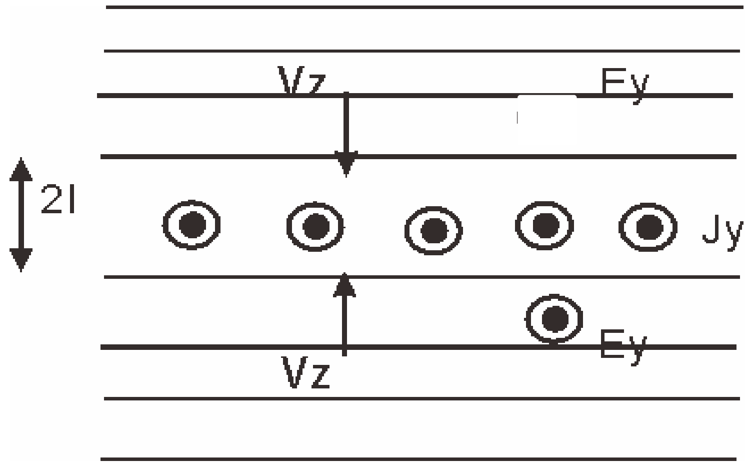
As the current sheet broadens, the opposite magnetic fields annihilate each other, which slows down the diffusion

Physics: Magnetic energy transformed to heat
Joule heating

Convection toward the current sheet



Let magnetic flux convect toward the current sheet to make the diffusion steady



Steady state $\partial \mathbf{B}/\partial t = 0$ → Faraday's law $\nabla \times \mathbf{E} = \frac{\partial E_y}{\partial z} = 0$

$\Rightarrow$ E_y is constant. Calculate it outside the current sheet: $E_y = VB_0$

At the current sheet $B = 0$ → Ohm's law: $\mathbf{E} = \mathbf{J}/\sigma \;\Rightarrow\; E_y = J_y/\sigma$

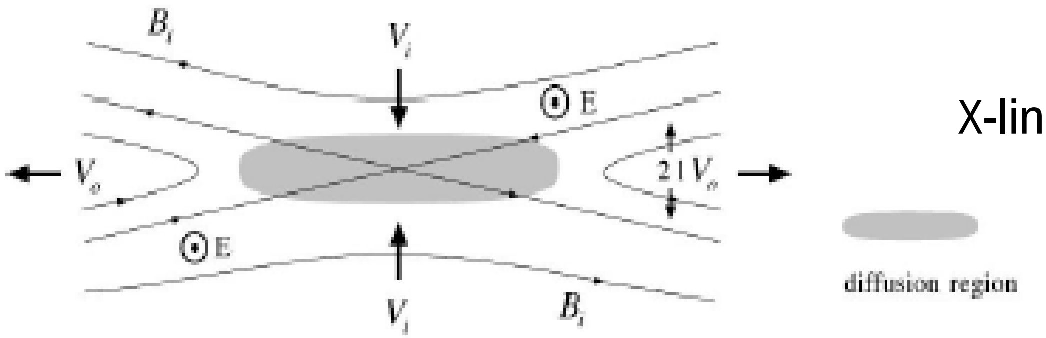
Let the thickness of the current sheet be $2l$.

From Ampère's law $J_y = \frac{B_0}{\mu_0 \cdot l}$ and thus $2l = \frac{2}{\mu_0 \sigma V}$ Rm ~ 1

Now the magnetic field is in steady-state, but what happens to plasma??

There is plasma inflow, but we have not provided any means for plasma to escape (Unphysical picture!)

Neutral line model of reconnection



X-line reconnection

Rm < 1

Idea: Add second dimension and assume that the annihilation takes place within a region limited in the x-direction

Plasma (and magnetic flux) inflow in z-direction
Plasma (and magnetic flux) outflow in x-direction
Magnetic neutral line in y-direction

Note that $\Phi_{out} < \Phi_{in}$, as the diffusion annihilates flux

Solar physics: Giovannelli 1940's
Magnetosphere: Sweet and Parker ~ 1958 (independent of each other)

Consequence

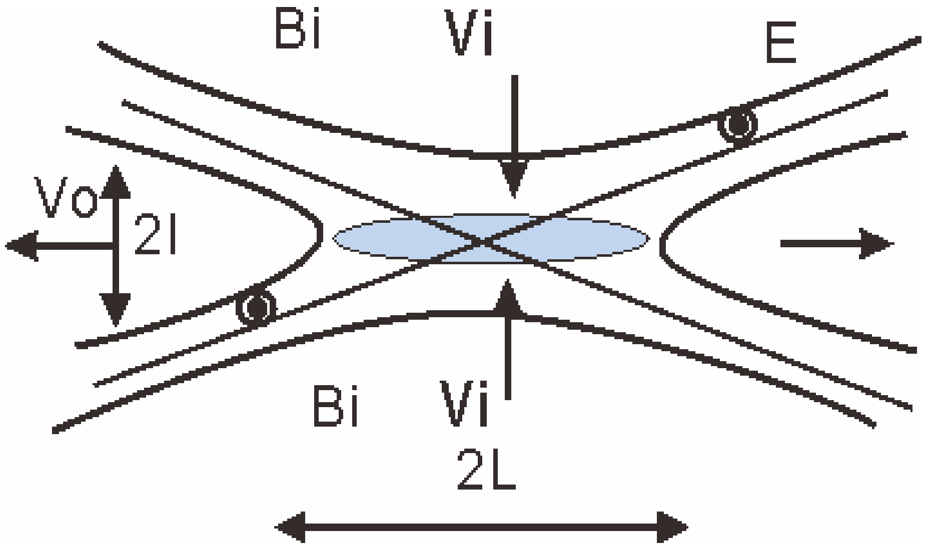
Magnetic field lines enter the diffusion region from the top to bottom, and instead of being annihilated, the leave from both sides. In the process, they are “cut” and “reconnected” to different partners.

Plasma now is free to move from one region to another, fundamentally changing the nature of the boundary.

Magnetic filed energy built up in the inflow regions is converted into heating and acceleration of the plasma in the outflow regions in the basic x-line picture.

The change in magnetic connection is the more profound effect, for this allows previously unconnected regions to exchange plasma readily, and hence mass, energy and momentum.

2D Sweet-Parker reconnection



Electric field constant

$$E = V_i B_i = V_o B_o$$

Incompressible flow

$$\rho_i = \rho_o = \rho$$

Conservation of mass

$$V_i L = V_o l$$

Inflow EM energy $|S| = |\mathbf{E} \times \mathbf{H}| = \frac{E B_i}{\mu_0} = \frac{V_i B_i^2}{\mu_0}$

The mass flowing in per unit area per unit time, ρV_i, is accelerated to speed V_o, so the rate of energy gain per unit area in the incident flow is $\Delta W = \frac{1}{2} \rho V_i (V_o^2 - V_i^2)$

Equate the kinetic energy gained by the outflowing plasma with the EM energy inflowing, and using Vo>>Vi

$$\Delta W = \frac{1}{2}\rho V_i (V_o^2 - V_i^2) = \frac{1}{2}\rho V_i V_o^2 = \frac{V_i B_i^2}{\mu_0}$$

$$V_o^2 = \frac{2B_i^2}{\mu_0 \rho} = 2V_{Ai}^2$$ Alfven velocity in the inflow region

The thickness of the diffusion region is $2l = \frac{2}{\mu_0 \sigma V}$

Inflow speed $V_i = V_{Ai} \left(\frac{\sqrt{2}}{R_{mA}}\right)^{1/2}$ Corresponds to the reconnection rate

where $R_{mA} = \mu_0 \sigma L V_A$ Reynolds number in the inflow region (Lundquist number)

In all solar-system plasma for which R_{mA} is very large, the inflow into the reconnection site, is Very Slow

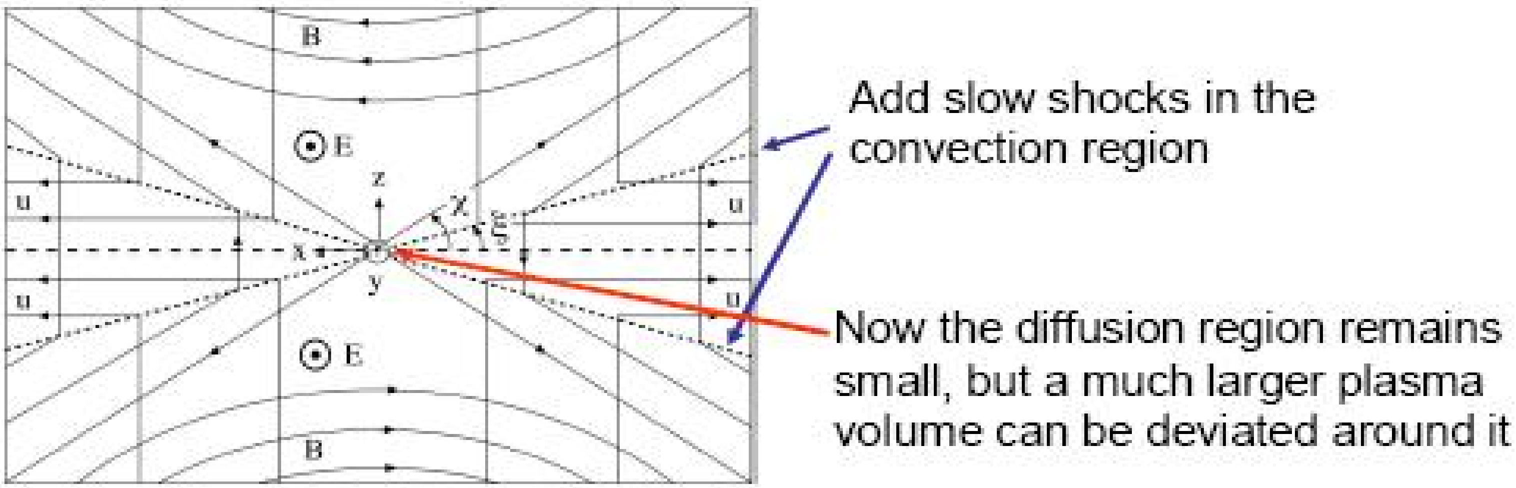
- Reconnection is described as diffusion on scales smaller than typical macroscopic scales.

- Properties in the inflow region (flow velocity, magnetic field strength, convection electric field) are related to the outflow region

- The dimensionless reconnection rate $\frac{V_i}{V_{Ai}} \sim \frac{1}{S^{1/2}}$, where S is the dimensionless Lundquist number.

- Reconnection rate is much slower as observed in space plasmas, but faster than global diffusion. For example, using typical solar corona parameters, a solar flare would take tens of days to grow, rather than a few minutes as observed.

How to make it faster: Petschek's model



The shocks are current sheets that accelerate plasma (**J** x **B** force)

Most of the plasma involved in the reconnection process does not need to flow through the diffusion region in order to be accelerated. Instead, it can be accelerated in the region where MHD is still valid.

The acceleration occurs as the plasma passes through shock waves due to the JxB force.

Petschek's model

In the frame of the figure, the shocks are stationary, but in the frame of the plasma the shocks travel along the magnetic field at the inflow Alfven speed.

The plasma inflow velocity normal to the shock must equal the shock speed normal to the shock front in the plasma frame for the shock to remain fixed. This gives

$$Vi \cos \xi = V_{Ai} \sin(\chi - \xi)$$

Again we can appeal to the steady state and hence require a uniform electric field in the y-direction.

$$E_i = u_i B_i \cos \chi = B_0 \cos \chi$$

Next, the component of B normal to the shock must be conserved.

$$B_i \sin(\chi - \xi) = B_0 \cos \xi$$

Petschek's model

Eliminating the magnetic field, gives:

$$V_o = \frac{V_i \cos\chi \cos\xi}{\sin(\chi - \xi)} = V_{Ai} \cos\chi$$

So, as in the Sweet-Parker solution, the outflow speed is comparable to the inflow Alfven speed. By conserving mass flow across the shock we get:

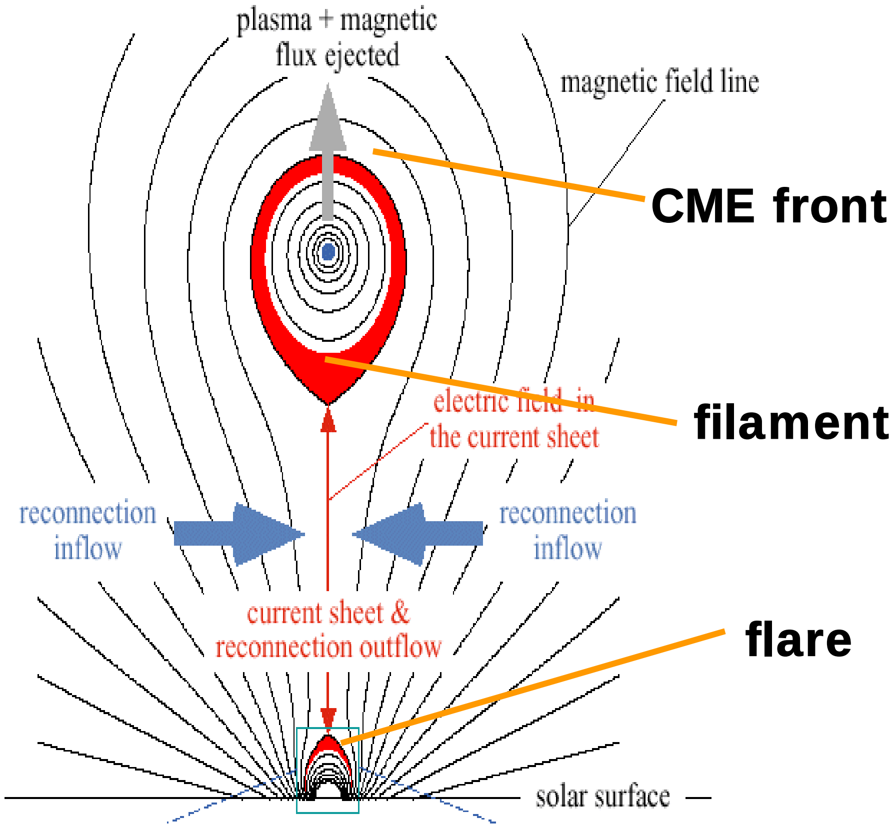
$$\rho_i V_i \cos\xi = \rho_o V_o \sin\xi$$

Where we have now allowed for the plasma to be compressed at the shock (i.e. $\rho_o \geq \rho_i$).

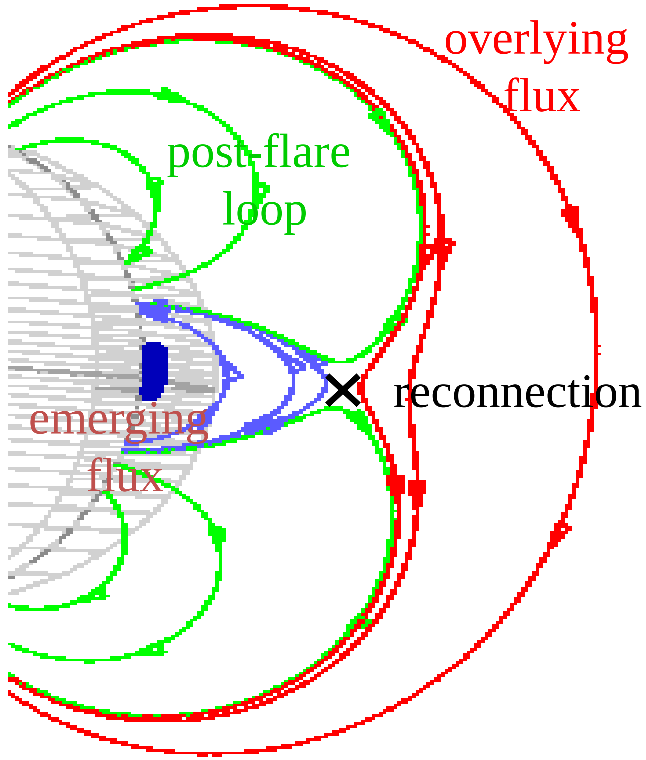
$$V_i = \frac{\rho_o V_{Ai} \cos\chi \sin\xi}{\rho_i \cos\xi}$$

So, in complete contrast to the Sweet-Parker solution, the inflow speed is some reasonable fraction of the inflow Alfven speed. However, further analysis of the constrains of the this solution shows $V_i \leq 0.1 V_{Ai}$

some CME-flare models

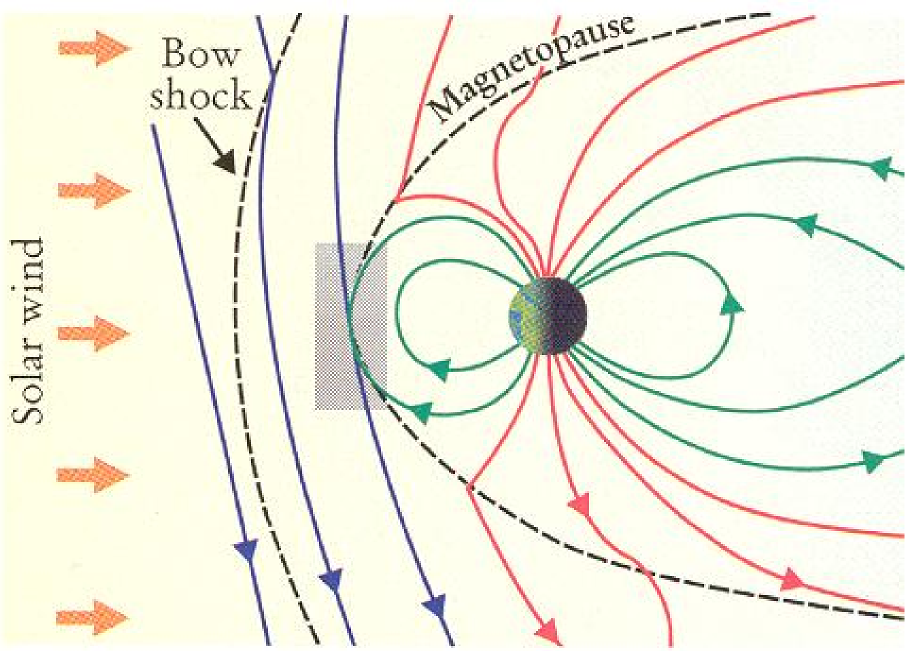


flux rope model
(Forbes-Priest-Lin)



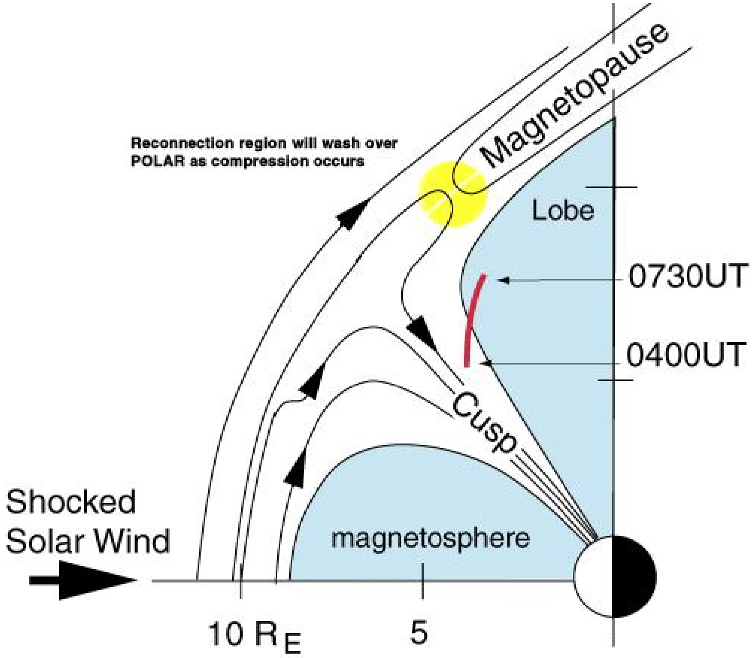
breakout model
(Antiochos et al. 1999)

Southward IMF



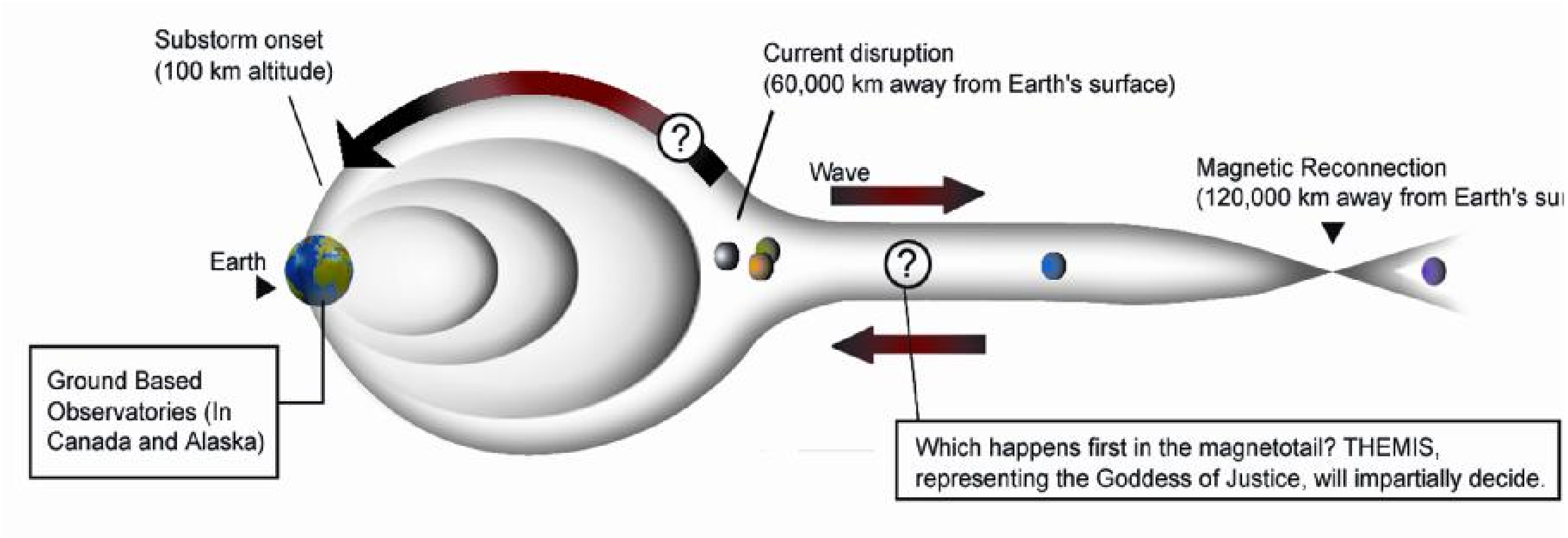
Adapted from Day, C., Spacecraft probes the site of magnetic reconnection in Earth's magnetotail, Physics Today, Vol. 54, No. 10, 16-17, 2001.

Northward IMF



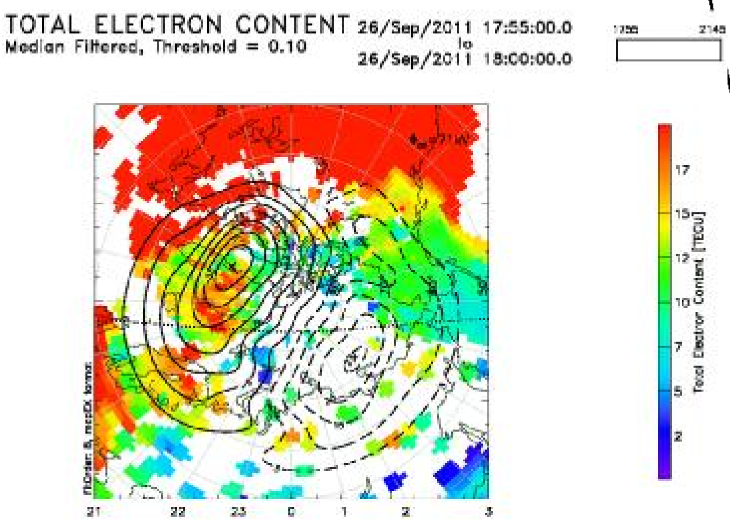
Adapted from NASA Press release: Connection of Sun's and Earth's magnetic fields provides energy for auroras, space weather, 2000.

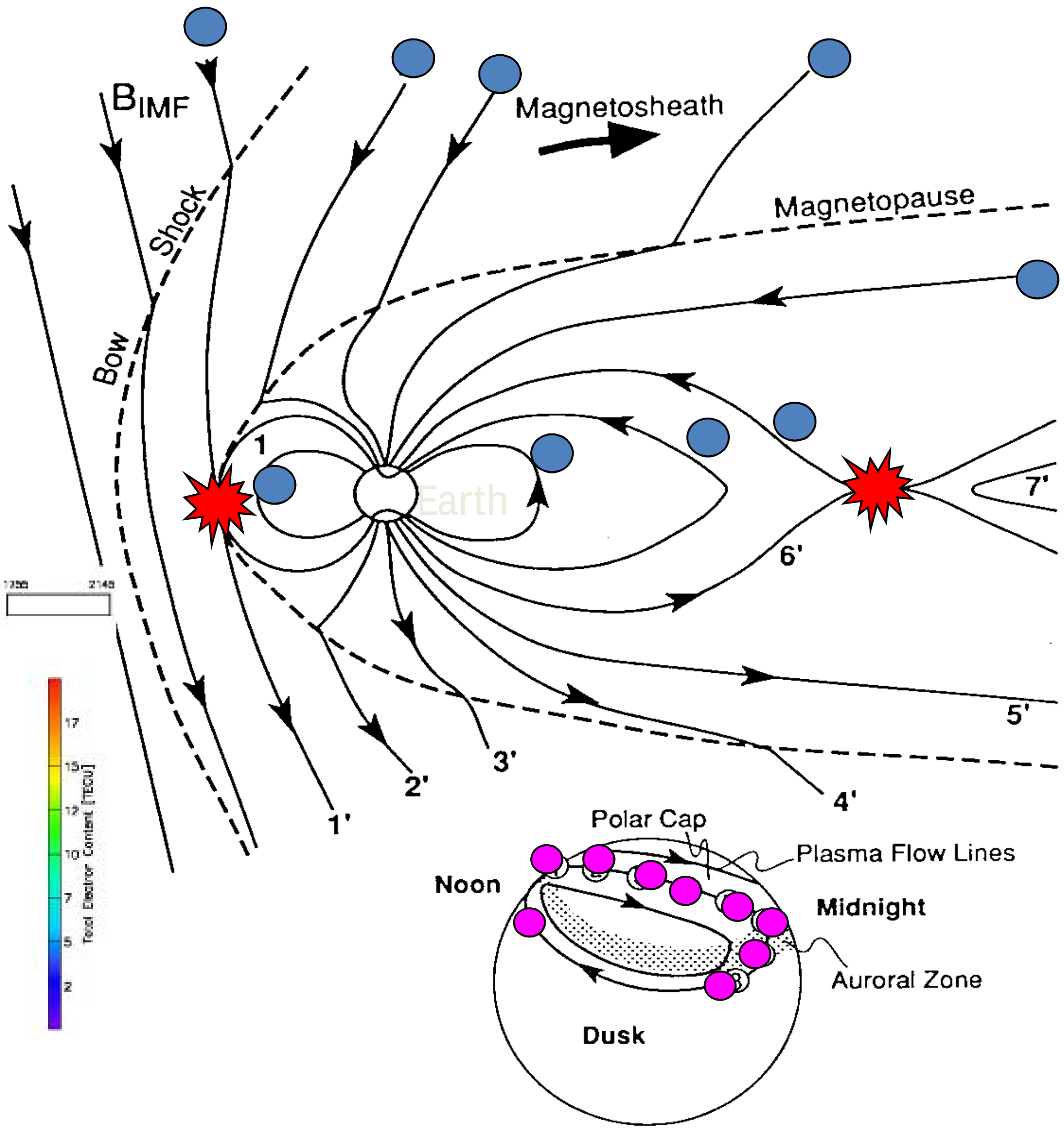
Magnetotail Reconnection



Dungey Cycle

Interplanetary field directed southward, which is antiparallel to the geomagnetic field.





Zhang, et al. Science, 2013

Magnetotail

As field lines are swept tailward in the solar wind, their ionospheric footprints are swept across the polar cap from near noon to near midnight. We can then estimate the time it takes plasma, and hence the foot of a field line, to cross the polar cap.

Assuming the polar cap has a radius of 15 degree or 1,500 km, and a plasma speed of 330 m/s, we obtain a time of 10^4 s or about 3 h.

The other ends of these polar-cap field lines are embedded in the solar wind flow, and so are moving away from the sun at speeds typically of 400 km/s. In 10^4 s, the move 4 x 10^6 km or about 600 Re.

This provides an estimate for the length of the part of the tail that is still connected to the earth magnetically, which is 10 times longer than its diameter.