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A New Hybrid Numerical Scheme for Two-Dimensional Ideal MHD Equations *

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We present a new hybrid numerical scheme for two-dimensional (2D) ideal magnetohydrodynamic (MHD) equations. A simple conservation element and solution element (CESE) method is used to calculate the flow variables, and the unknown first-order spatial derivatives involved in the CESE method are computed with a finite volume scheme that uses the solution of the derivative Riemann problem with limited reconstruction to evaluate the numerical flux at cell interface position. To show the validation and capacity of its application to 2D MHD problems, we study several benchmark problems. Numerical results verify that the hybrid scheme not only performs well, but also can retain the solution quality even if the Courant number ranges from close to 1 to less than 0.01.

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The space-time conservation element and solution element (CESE) method, originally proposed by Chang,[1] and Chang, Wang and Chow,[2,3] is a powerful numerical frame for solving conservation laws. This method is non-conventional by differing substantially from other well-established finite difference methods. To date the CESE method has seen its great success in studying flows with moving and steady shocks, acoustic waves, complex vertical flows, etc.[2−8]

The original CESE scheme[3] has been extended by Zhang *et al.*[9] for the numerical solution of the ideal magnetohydrodynamics (MHD) equations. However, the numerical dissipation associated with the CESE scheme when using a fixed total marching time increases as the Courant number (CFL) decreases. As a result, for a small CFL number (say <0.1), a CESE scheme may become overly dissipative. In the CESE scheme, the solution variables are left as unknowns, and their spatial derivatives are obtained with the finite-difference-based approach or the central-difference like weighted average approach.[5−7] This results in the introduction of some numerical dissipation into the system. From this point, we find that obtaining the derivatives is key to improving the CESE method.

In this Letter, we propose a new approach instead of using the routine weighted average approach to compute the spatial derivatives. The new procedure for calculating the spatial derivatives, combined with the simple CESE scheme for calculating solution variables, forms our new hybrid scheme of the present study. In computing these spatial derivatives, we evaluate the numerical flux at the interface by solving the derivative Riemann problem (DRP). The solution of the DRP has been used to structure ADER (arbitrary accuracy derivative Riemann problem) methods. The ADER approach for constructing high order methods was first put forward by Toro and collaborators for linear problems on Cartesian meshes.[10] At present, the ADER approach has been applicable to multidimensional nonlinear systems.[11,12]

In the following, we will briefly describe the new hybrid method. The two-dimensional ideal MHD equations can be cast into the following conservative form:

$$\frac{\partial \boldsymbol{U}}{\partial t}+\frac{\partial \boldsymbol{F}(\boldsymbol{U})}{\partial x}+\frac{\partial \boldsymbol{G}(\boldsymbol{U})}{\partial y}=0, \tag{1}$$

where $\boldsymbol{U}=(\rho,\rho u,\rho v,\rho w,e,B_x,B_y,B_z)^T$ is the vector of conserved variables, $\boldsymbol{F}$ and $\boldsymbol{G}$ are the conservation flux vectors in x and y directions, respectively. Here ρ and p are the density and gas pressure, respectively; $\boldsymbol{u}=(u,v,w)$ and $\boldsymbol{B}=(B_x,B_y,B_z)$ are velocity components and magnetic field components in the x,y,z directions, respectively. The specific total energy e is $e=p/(\gamma-1)+\rho\boldsymbol{u}^2/2+\boldsymbol{B}^2/2$.

Due to space limitations, we will not introduce in detail the CESE method for calculating the flow variables based on the regular quadrilateral meshes in two-dimensional space. We divide the x–y plane into non-overlapping uniform quadrilaterals and any two neighboring quadrilaterals share a common side (Fig. 1(a)). The centroids of quadrilaterals are marked by hollow circles. Point Q is the centroid of a typical quadrilateral $B_1B_2B_3B_4$, and is also the centroid of polygon $A_1B_1A_2B_2A_3B_3A_4B_4$, which coincides with quadrilateral $A_1A_2A_3A_4$. The points $A_\ell, \ell=1,2,3,4$, respectively, are the centroids of the four quadrilaterals neighboring to the quadrilateral $B_1B_2B_3B_4$. The definition of the conservation element (CE) and the solution element (SE) (see Fig. 1(b)) follows that of

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Qamar and Mudasser.[8] The solution of the flow variables can be obtained by solving the MHD equations on CE and SE above, which is the same as Eq. (20) in Ref. [8]. For a more detailed derivation, the reader can refer to Zhang *et al.*[5]

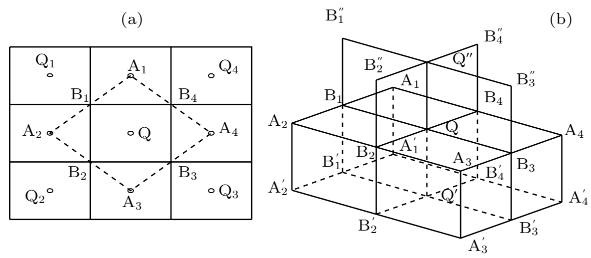

Fig. 1. Space-time geometry of the CESE method: (a) representative grid points in the x–y plane, (b) the definitions of CE and SE.

If all flow variables and their spatial derivatives at the $n-1/2$ time level are known, we can calculate flow variables at the n new time level. Here n is the index for t. However, for the calculation at the $n+1/2$ time level, we still need their spatial derivatives at the n new time level. Formerly, in a CESE scheme, the spatial derivatives at the n new time level are calculated through several finite-difference-based approaches and the central-difference like weighted average approach.[5–7]

In this study, we resort to the DRP method for updating the first order spatial derivatives. Motivated by DRP used in the ADER method,[11,12] we first construct the evolution equations for the spatial derivatives, then use the unsplit finite volume approach to calculate the corresponding evolution equations of these spatial derivatives.

We first apply the chain rule to Eq. (1) and obtain

$$\partial_t(\boldsymbol{U}) + J^x\partial_x(\boldsymbol{U}) + J^y\partial_y(\boldsymbol{U}) = 0, \tag{2}$$

where J^x and J^y are Jacobian matrices. Then we construct the evolution equations for spatial derivatives by differentiating Eq. (2) as follows:

$$\begin{aligned}&\partial_t(\partial_x\boldsymbol{U}) + \partial_x(J^x\partial_x\boldsymbol{U}) + \partial_x(J^y\partial_y\boldsymbol{U}) = 0,\\ &\partial_t(\partial_y\boldsymbol{U}) + \partial_y(J^x\partial_x\boldsymbol{U}) + \partial_y(J^y\partial_y\boldsymbol{U}) = 0.\end{aligned} \tag{3}$$

Owing to $\partial_x(J^y\partial_y\boldsymbol{U}) = \partial_y(J^y\partial_x\boldsymbol{U})$ and $\partial_y(J^x\partial_x\boldsymbol{U}) = \partial_x(J^x\partial_y\boldsymbol{U})$ when J^x and J^y do not depend on x and y, Eq. (3) can be written in the form

$$\partial_t(\mathcal{F}) + \partial_x(J^x\mathcal{F}) + \partial_y(J^y\mathcal{F}) = 0, \tag{4}$$

where $\mathcal{F}$ stands for $\partial_x\boldsymbol{U}$ or $\partial_y\boldsymbol{U}$.

Using Gauss's law, the integration form of Eq. (4) can be written as

$$\frac{\partial}{\partial t}\int \mathcal{F}dv + \int (J^x\mathcal{F}n_x + J^y\mathcal{F}n_y)ds = 0, \tag{5}$$

where dv and ds are the volume and surface elements of the control volume, and $\boldsymbol{n}$ is the unit vector normal to the surface of the control volume. For the two-dimensional case, we have

$$J^x\mathcal{F}n_x + J^y\mathcal{F}n_y = \boldsymbol{T}^{-1}J^x_{\boldsymbol{n}}\mathcal{F}_{\boldsymbol{n}}, \tag{6}$$

where $\boldsymbol{T}^{-1}$ is the inverse of the rotation matrix $\boldsymbol{T}$, which rotates the x axis to the direction of $\boldsymbol{n}$.

We consider a typical quadrilateral finite volume $V_Q = A_1A_2A_3A_4$ of a two-dimensional domain, as depicted in Fig. 1(a). The finite volume has four intercell boundaries. From Eqs. (5) and (6), a discrete formulation of the evolution equation in the finite volume method style for the grid point Q is written in the form

$$\frac{\partial}{\partial t}\mathcal{F}_Q S + \sum_{\ell=1}^{4}\boldsymbol{T}_\ell^{-1}(J^x_{\boldsymbol{n}_\ell}\mathcal{F}_{\boldsymbol{n}_\ell})\lambda_\ell = 0, \tag{7}$$

where ℓ denotes the number of the grid points to neighboring grid point Q, S denotes the area of the control volume cell containing grid point Q, and λ_ℓ is the length of the ℓth side face.

In order to define numerical fluxes across the intercell boundaries, we solve Riemann problems for spatial derivatives (DRP) in the direction normal to the cell edge coupled with limited linear reconstruction. As the above evolution equation (4) is too complicated, following the simplified approach of Toro and Titarev,[13] we also assume the equation to be linear with a constant coefficient matrix $J^x_{LR} = J^x(U(0,0_+))$, where $U(0,0_+)$ can be obtained by solving the classical Riemann problem.[11,13] Then we obtain the following *linearized* and *classical* Riemann problems,

$$\begin{aligned}&\partial_t(\mathcal{F}(x,t)) + J^x_{LR}\partial_x(\mathcal{F}(x,t)) = 0,\\ &\mathcal{F}(x,0) = \begin{cases}\mathcal{F}_L(0) & \text{if} \quad x<0,\\ \mathcal{F}_R(0) & \text{if} \quad x>0,\end{cases}\end{aligned} \tag{8}$$

where $\mathcal{F}_L$ and $\mathcal{F}_R$ are the reconstructions of $\mathcal{F}$ on the left and right sides of interface, respectively. Then the interface fluxes can use the upwind flux, for example, the Roe flux.[14]

In order for the scheme to be more than first-order accurate, a local reconstruction must be carried out; in order to damp off numerical oscillations, the reconstruction must be limited. For a given cell with center Q, the linear reconstruction of the spatial derivatives is limited in the form

$$\mathcal{F}_m(x,y) = \mathcal{F}_m + \varphi_m\nabla\mathcal{F}_m\cdot\boldsymbol{r}, \tag{9}$$

where $m = 1, 2\cdots 8$, $\mathcal{F}_m$ is the cell-averaged value prescribed at Q, φ_m is a chosen limiter, $\boldsymbol{r}$ is the vector extending from the cell center Q to any point (x,y) within the cell, and $\nabla\mathcal{F}_m$ is the cell-centered gradient. In a manner similar to those used in Ref. [14], the limiter is defined as

$$\begin{aligned}\varphi_m = \min\Big(1, &\frac{|\mathcal{F}_m - \max_{neighbors}(\mathcal{F}_m)|}{|\mathcal{F}_m - \max_{cell}(\mathcal{F}_m)|},\\ &\frac{|\mathcal{F}_m - \min_{neighbors}(\mathcal{F}_m)|}{|\mathcal{F}_m - \min_{cell}(\mathcal{F}_m)|}\Big),\end{aligned} \tag{10}$$

where the subscript *neighbor* denotes the neighboring cells used in the gradient reconstruction, and the subscript *cell* denotes the unlimited ($\varphi = 1$) reconstruction to the centroids of the faces of the cell. Now, $\mathcal{F}_L$ and $\mathcal{F}_R$ can be computed by taking the end point of the vector $\boldsymbol{r}$ in Eq. (9) as the midpoint of the faces of the cell. Finally, The combination of the CESE method for calculating the flow variables and Eq. (7) constitutes the new hybrid scheme.

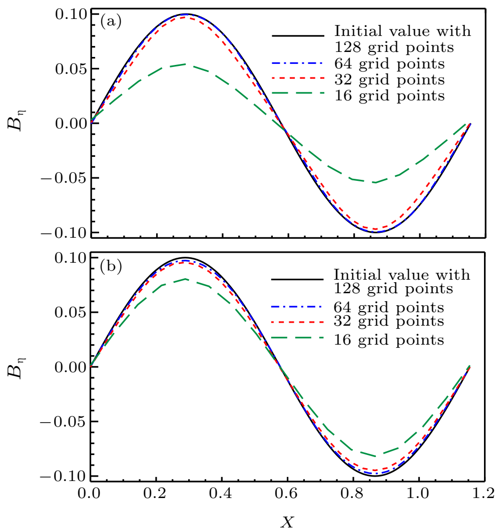

Fig. 2. Distribution of B_η with different mesh resolution at $t = 5$ (a) for the traveling Alfvěn ($v_\xi = 0$, $v_A = -1$) wave problem, (b) for the standing Alfvěn ($v_\xi + v_A = 0$) wave problem.

Table 1. The average errors and the orders of accuracy at $t = 5$.

N	Traveling waves		Standing waves	
	L_1 Error	L_1 order	L_1 Error	L_1 order
16	3.17×10^{-2}		1.38×10^{-2}	
32	5.38×10^{-3}	2.5588	2.94×10^{-3}	2.2308
64	9.27×10^{-4}	2.5370	7.81×10^{-4}	1.9124

To show the validity of the hybrid scheme, we simulate several benchmark test cases. The smooth Alfvěn wave problem was suggested in Toth[15] as a test for numerical accuracy of the scheme for smooth flow. The Alfvěn wave propagates at an angle of $\alpha = 30^\circ$ with respect to the x axis in the domain $[0, 1/\cos\alpha] \times [0, 1/\sin\alpha]$. The initial conditions are $\rho = 1$, $v_\xi = 0$, $v_\eta = B_\eta = 0.1\sin(2\pi\xi)$, $v_z = B_z = 0.1\cos(2\pi\xi)$, $B_\xi = 1$, $p = 0.1$ with $\gamma = 5/3$, where $\xi = x\cos\alpha + y\sin\alpha$, and $\eta = y\cos\alpha - x\sin\alpha$. The Alfvěn wave is a traveling wave. Note that the wave becomes standing if $v_\xi = 1$.

The problem is solved on a set of rectangular $N \times 2N$ meshes with $N = 16, 32, 64$. The numerical error of variable u is calculated in an L_1 norm defined as $\delta_N u = \frac{1}{N\times 2N}\sum_{i,j}|u_{i,j} - u_{i,j}^{exact}|$.. An averaged value is computed as $\frac{1}{4}(\delta_N(v_\eta)+\delta_N(v_z)+\delta_N(B_\eta)+\delta_N(B_z))$. Periodic boundary conditions are imposed in both the x and y directions. The simulation is run to a final time $t = 5$ with CFL = 0.8. Figure 2 shows the profile of B_η along the line of $y = 0$ with a different mesh resolution for the traveling wave problem and the standing wave problem. The errors in the wave amplitude are quickly reduced with the use of a refined mesh. The solution obtained by the mesh of 64×128 is nearly identical to the analytical solution. Table 1 gives the average numerical errors and orders of accuracy obtained by the hybrid scheme. The results show that the new hybrid scheme converges approximately at a second order rate for smooth solutions.

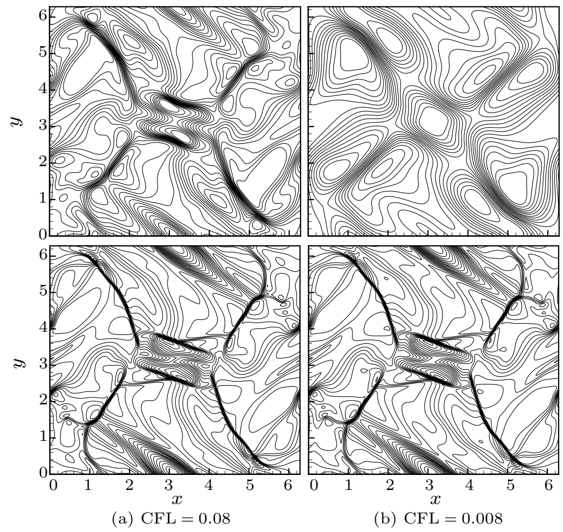

Fig. 3. Solution comparison of MHD Vortex problem by using the CESE scheme (top) and the hybrid scheme (bottom) at $t = 3.0$ with CFL=0.08 (a) and CFL=0.008 (b).

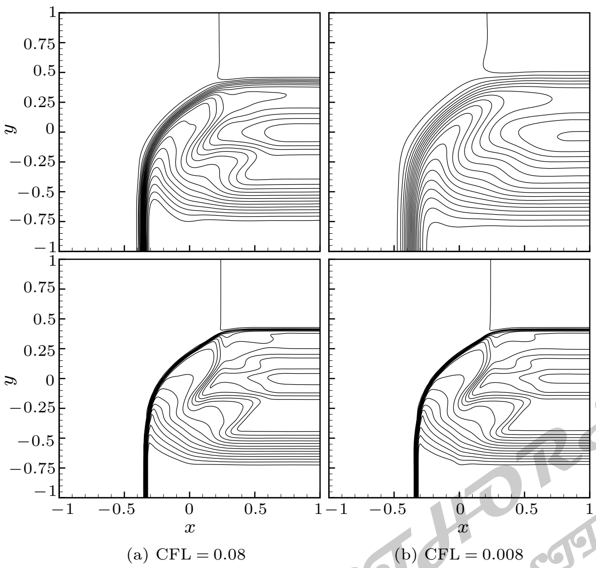

Fig. 4. Contour plots for MHD Riemann problem with CFL=0.08 (a) and with CFL=0.008 (b) at $t = 0.2$. Upper: CESE scheme. Lower: hybrid scheme.

The MHD vortex problem has been studied by many previous investigators.[7–9] Although not shown, the results calculated using the hybrid scheme with CFL = 0.8 at $t = 0.5$, 2 and 3, respectively, are almost identical to those of Zhang *et al.*[9] Here we mainly compare the solution quality of the CESE scheme and the hybrid scheme when CFL is less than 0.1. Figure 3 shows the solution comparison of the density with CFL = 0.08 and CFL = 0.008 at $t = 3.0$ by using the two schemes. From the results shown, it is clear that the CESE solution deteriorates quickly and does not capture the shock effectively as the value of global CFL number drops below 0.1. The hybrid scheme can still capture the shock effectively even if the value of global CFL number is less than 0.01. The advantage of the hybrid scheme over the CESE scheme is obvious.

We also simultaneously solve the Riemann problem with different CFL numbers. Figure 4 shows the solution comparison of the density with CFL = 0.08 and CFL = 0.008 at $t = 0.2$ using the two schemes, respectively. As before, we can see that the CESE scheme is more dissipative than the hybrid scheme when the CFL number is less than 0.1. The CESE scheme does not capture the shock effectively, but the hybrid scheme can capture it. It also further approves that the hybrid scheme can capture the shock effectively even if CFL is less than 0.01.

In summary, we have presented a new hybrid method for solving 2D ideal MHD equations. To demonstrate the capabilities of the hybrid method, three benchmark problems are calculated. The results of smooth Alfvěn wave problem indicate that for smooth flows the present scheme is second-order accurate. By testing different CFL numbers, we have found that the new scheme can retain the solution quality even if the CFL number ranges from close to 1 to less than 0.01. This feature mitigates the numerical dissipation caused by a small local CFL condition, which is formerly overcome by the Courant number insensitive scheme.[17,18] Their scheme also decreases numerical dissipation by improving the procedure for calculating the spatial derivatives, but the modified procedure is still an essentially central difference like the weighted average approach. Our new method for computing the spatial derivatives is different from the former approach through solving the corresponding time-dependent equations of the spatial derivatives, which obey the original system of MHD equations and can also be calculated by other numerical schemes, such as: the Roe method and Godunov-type methods. The method of calculating derivatives is a completely new way. However, due to the DRP calculation, the new hybrid scheme incurs extra computational cost. However, this study shows that the proposed method is a significant improvement of the CESE method and very suitable for solving MHD equations. Our future aim is to use the new hybrid method to analyse the solar wind problem in interplanetary space since a large disparity of CFL numbers exists from the Sun to Earth space.[19]

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