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New approach for solving the inverse boundary value problem of Laplace's equation on a circle: Technique renovation of the Grad-Shafranov (GS) reconstruction

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[1] In this paper, the essential technique of Grad-Shafranov (GS) reconstruction is reformulated into an inverse boundary value problems (IBVPs) for Laplace's equation on a circle by introducing a Hilbert transform between the normal and tangent component of the boundary gradients. It is proved that the specified IBVPs have unique solution, given the known Dirichlet and Neumann conditions on certain arc. Even when the arc is reduced to only one point on the circle, it can be inferred logically that the unique solution still exists there on the remaining circle. New solution approach for the specified IBVP is suggested with the help of the introduced Hilbert transform. An iterated Tikhonov regularization scheme is applied to deal with the ill-posed linear operators appearing in the discretization of the new approach. Numerical experiments highlight the efficiency and robustness of the proposed method. According to linearity of the elliptic operator in GS equation, its solution can be divided into two parts. One is solved from a semilinear elliptic equation with an homogeneous Dirichlet boundary condition. The other is solved from the IBVP of Laplace's equation. It is concluded that there exists a unique solution for the so-called elliptic Cauchy problem for the essential technique of GS reconstruction.

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1. Introduction

[2] As in spacecraft data analysis, there is a novel tool called Grad-Shafranov (GS) reconstruction to recover the steady two-dimensional coherent magnetic structures from observational data along paths of the spacecraft. The essential idea of GS reconstruction technique, i.e., discretizing the Laplace's operator of the GS equation with a finite difference scheme, and producing the Cauchy data from observations along the path, and then solving it as a closed system on a narrow strip around the path, is first introduced by *Sonnerup and Guo* [1996] and then renovated by *Hau and Sonnerup* [1999] with a weighted three-point smoothing approach to suppress the exponential increase of the numerical errors (for details, see a recent review of *Lui* [2011] and the handbook from *Möstl and Farrugia* [2010]). This technique is later implemented by *Hu and Sonnerup* [2001] and then is applied to study the interplanetary magnetic cloud structures [*Hu and Sonnerup*, 2001; *Hu et al.*, 2003]. Recently, the technique has been improved further by *Isavnin et al.* [2011], where the filtering approach of *Hau and Sonnerup* [1999] is replaced by a digital differentiator. Efforts in designing and updating the filtering scheme for GS technique are in fact to find a proper way to settle the ill-posedness contained in its solution approach, which is the main mathematic obstacle confronted by the GS reconstruction. In this investigation, we try to settle the ill-posedness with the boundary integrals of the elliptic operators, and thus, the essential idea of the GS reconstruction technique is considered as the IBVPs of Laplace's equation on a circle.

[3] Conception of "inverse boundary value problems (IBVPs) for Laplace's equation" can be retrospected to about 110 years ago [*Hadamard*, 1902], when *Hadamard* laid the basis of the concept of well-posed problems, and used the "Cauchy problem for the Laplace's equation" as his first example of a problem that is not well-posed. Later on, in 1923, he published his well-known example of instability and provided a fundamental example that shows a solution of a Cauchy problem for Laplace's equation does not depend continuously upon the data [*Hadamard*, 1923]. From then on, the so-called "Cauchy problem for Laplace's equation" has remained as a typical ill-posed problem or an inverse problem and undergone an intensive study.

[4] The Cauchy problem for Laplace's equation is proved to be severely ill-posed; that is, any small change in the

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initial data may result in a dramatic change in the solution. Theoretical analysis of *Belgacem* [2007], investigated the severe ill-posedness of the Cauchy problem, by using a Steklov-Poincaré approach. The existence and uniqueness of a weak solutions for arbitrary Cauchy data was investigated by *Calderón* [1958] and *Engl and Leitão* [2001]. It has been proved that a continuous dependence of the solution on the initial data can be obtained under an additional a priori boundedness condition [*Payne*, 1960, 1970; *Tautenhahn*, 1990; *Reinhardt et al.*, 1999; *Alessandrini et al.*, 2009], which is called the conditional stability.

[5] In despite of the great difficulty to achieve the right solution due to its high ill-posedness, a variety of numerical methods are developed under the inspiration of theoretical results, e.g., the quasi-reversibility method [*Klibanov and Santosa*, 2007; *Bourgeois*, 2005, 2006, 2010], the Backus-Gilbert method [*Cheng et al.*, 2001; *Hon and Wei*, 2001], the Mann iterative regularization method [*Engl and Leitão*, 2001], the conjugate gradient method [*Hào and Lesnic*, 2000], the optimizational method [*Kabanikhin and Karchevsky*, 1995], and the level set method [*Leitão and Alves*, 2007]. Although details for each method may be very different, they have a common character, namely, an additional constraint on the discrete method, or the regularization parameters are needed to satisfy the so-called stability condition.

[6] Controversies arise from the treatment of its ill-posedness. It is argued that using the term "elliptic Cauchy problem" is inappropriate, since both the analytic and numerical solution to a partial differential equation (PDE) largely depend on the specific type of the PDE. The solution of an elliptic PDE depends on an enclosed boundary (condition), whereas signals of a hyperbolic PDE for an initial value problem propagate along the characteristics [e.g., *Courant and Hilbert*, 1962]. As for the Cauchy problem or the proof of the Cauchy theorem, everything (such as "domain of influence") associated with the solution is determined by the characteristics. On the other hand, an elliptic PDE has no characteristics, or the eigenvalues of the characteristic equation are purely imaginary.

[7] We prefer to use the term of inverse boundary problem to the Cauchy one, since the considered problem is essentially an elliptic problem and is well-posed. Thus, we need only to recover the missing boundary data, which is of an inverse problem on the Sobolev space [*Yu*, 2006], and this inverse problem may be or may not be an ill-posed one, which depends upon the known information. This treatment is different from the philosophy of the elliptic Cauchy problem, where the ill-posedness is treated integrally throughout its solution procedure.

[8] As for the essential technique of GS reconstruction, we concluded that there exists an unique solution for the so-called elliptic Cauchy problem. According to linearity of the elliptic operator in GS equation, this solution can be divided into two parts, one is solved from a semilinear elliptic equation with an homogeneous Dirichlet boundary condition, and the other is solved from the IBVP of the Laplace's equation. Solution approaches for the former problem are discussed maturely in literatures, which is of a trivial task, so that there is only nontrivial task left, that is, to find an approach most suitable for the solution of the IBVP.

[9] In this paper, we focus on the problem of missing data completion for Laplace's equation on a circle, which aims at recovering missing conditions on some inaccessible part of the boundary from the over-specified boundary data on the remaining part. There are several numerical approaches in the literature following this idea, e.g., *Cao and Pereverzev* [2007], *Chapko and Johansson* [2009], *Gupta* [2012], *Kozlov et al.* [1991], and *Tajani and Abouchabaka* [2012a, 2012b], although most of them are still entitled with the elliptic Cauchy problem. A new solution approach are developed following an idea of reproducing kernel of Hilbert space [*Takeuchi and Yamamoto*, 2008]. In contrast to those approaches, we find that the Hilbert transform exists between components of the gradients on a circle, and with the help of this Hilbert transform, we develop a new solution approach. We first discuss the Hilbert transform in section 2 and introduce the new approach for IBVPs in section 3. Cases of benchmark tests and a summary are presented in sections 4 and 5, respectively.

2. Hilbert Transform for BVPs of Laplace's Equation on a Circle

[10] Consider the Dirichlet problem of the Laplace's equation over a plane circular domain $\Omega \subset \mathbf{R}^2$ with a piecewise smooth boundary Γ,

$$\begin{cases} \triangle u = 0 & \text{in } \Omega \\ u = u_0 & \text{on } \Gamma, \end{cases} \tag{1}$$

and the Neumann problem,

$$\begin{cases} \triangle u = 0 & \text{in } \Omega \\ u_n = g & \text{on } \Gamma, \end{cases} \tag{2}$$

where $\Gamma = \partial\Omega$, $u_n = \frac{\partial u}{\partial n}$ and n is the normal to Γ, oriented toward the exterior of Ω. For the BVP (1), there exists a unique solution, and for problem (2), there exists a unique solution up to an additive constant when the compatibility condition,

$$\int_\Gamma g \mathrm{d}s = 0, \tag{3}$$

is satisfied.

[11] Component of the boundary gradients $\nabla u|_\Gamma = (u_t, u_n)$ can be uniquely determined from the solution of BVPs (1, 2), e.g., for the BVP (1), $u_t = \frac{\mathrm{d}}{\mathrm{d}s} u_0(s)$ is the tangent derivative of the known Dirichlet condition. As $s \in \Gamma$, $u_t(s)$ satisfies the periodic condition:

$$\int_\Gamma u_t(s)\mathrm{d}s = 0, \tag{4}$$

and u_n can be determined from its unique solution; for the BVP (2), $u_n = g$ is known; under the compatibility condition (3), u_t can be determined from its unique solution.

[12] If the constraint conditions of (3) and (4) are satisfied, we find that there exists an important relation between the component pair (u_t, u_n), that is, the Hilbert transform:

$$u_n(x) = \frac{1}{2\pi} \int_\Gamma \cot \frac{x - s}{2} u_t(s)\mathrm{d}s, \quad x \in \Gamma, \tag{5}$$

$$u_t(s) = \frac{1}{2\pi} \int_\Gamma \cot \frac{s - x}{2} u_n(x)\mathrm{d}x, \quad s \in \Gamma, \tag{6}$$

where $⨍$ denote the Cauchy principal integral, and items at the right of equations (5) and (6) are singular integrals with an Hilbert kernel.

[13] Theories of BVPs for Elliptic equations tell us that if we want to get (u_t, u_n), we need first to solve the BVP (1) or (2) with one of the known conditions u_t or u_n under corresponding constraints (3) or (4). But with the help of the Hilbert transform in equations (5) and (6), it does not need to solve the BVPs at all. One component of the boundary gradients can be determined analytically from the other one through the dual singular boundary integrals. The relations in equations (5) and (6) is of great value in practice but has seldom been discussed in literatures to our knowledge. Detailed derivation of these relations are out of the scope of this paper and will be presented elsewhere.

3. New Approach for Solving IBVPs Based on the Hilbert Transform

[14] The IBVPs of the Laplace's equation over a plane circular domain $\Omega \subset \mathbf{R}^2$ with a piecewise smooth boundary ($\Gamma = \partial\Omega$) are considered here. We denote the IBVPs as follows

$$\begin{cases} \triangle u = 0 & \text{in} \quad \Omega \\ u = u_0 & \text{on} \quad \Gamma_0 \\ u_n = g_0 & \text{on} \quad \Gamma_0, \end{cases} \tag{7}$$

where (u_0, g_0) are the known data given on $\Gamma_0 (\Gamma_0 \cup \Gamma_1 = \Gamma$ and $\Gamma_0 \cap \Gamma_1 = \emptyset)$. The new approach is developed to infer the unknown information on Γ_1, under the assumption of $⨍_\Gamma u_t(s)\mathrm{d}s = 0$ and $⨍_\Gamma u_n(x)\mathrm{d}x = 0$.

3.1. Existence and Uniqueness of the IBVPs

[15] The existence and uniqueness of the mixed boundary problem for elliptic equation have been proved by *Gupta and Cao* [2009]. We present a proof of the existence and uniqueness of the IBVP (7) in this section with the help of a corollary derived from a theory of *Gupta and Cao* [2009] and the introduced Hilbert transform.

[16] *Lemma* 3.1
(Theorem 3.1, section 3) *Gupta and Cao* [2009] For every $f \in L^2(\Omega)$, every $\phi \in L^2(\Gamma_3)$, and every $g_1 \in H^{-1/2}(\Gamma_0)$, the system of equations (3.5) admits a unique solution $u(f, \phi, g_1)$ in $L^2(\Omega)$.

[17] *Corollary* 3.1
For every $u_0 \in L^2(\Gamma_0)$ and every $g_1 \in H^{-1/2}(\Gamma_1)$, the system of equations,

$$\begin{cases} \triangle u = 0 & \text{in} \quad \Omega \\ u = u_0 & \text{on} \quad \Gamma_0 \\ u_n = g_1 & \text{on} \quad \Gamma_1, \end{cases} \tag{8}$$

admits a unique solution $u(u_0, g_1)$ in $L^2(\Omega)$, where $(\Gamma_0 \cup \Gamma_1 = \Gamma$ and $\Gamma_0 \cap \Gamma_1 = \emptyset)$.

[18] *Theorem* 3.1
For every $u_0 \in L^2(\Gamma_0)$ and every $g_0 \in H^{-1/2}(\Gamma_0)$, the systems of IBVP (7) admits a unique solution for boundary gradients on the inaccessible part of Γ_1, e.g., (u_{t1}, u_{n1}) in $H^{-1/2}(\Gamma_1)$.

[19] **Proof.** (1) Let us assume that another solution u^*_{n1} exists in $H^{-1/2}(\Gamma_1)$ for the IBVP (7) besides u_{n1}. With the Corollary 3.1, we assure that there exists corresponding unique solution for the mixed BVPs (8), say $u(u_0, u_{n1})$ and $u^*\left(u_0, u^*_{n1}\right)$ in $L^2(\Omega)$.

[20] Obviously, $\gamma_0 u|_{\Gamma_0} = \gamma_0 u^*|_{\Gamma_0} = u_0$, where γ_0 is the Dirichlet trace operator. The enclosed boundary derivatives for the tangent component can be denoted by $u_t = \frac{\mathrm{d}}{\mathrm{d}s}\gamma_0 u$ and $u^*_t = \frac{\mathrm{d}}{\mathrm{d}s}\gamma_0 u^*$, respectively. Since $\gamma_0 u$ and $\gamma_0 u^*$ are in $H^{-1/2}(\Gamma)$, the periodic constraint are satisfied, i.e., $⨍_\Gamma u_t(s)\mathrm{d}s = 0$ and $⨍_\Gamma u^*_t(s)\mathrm{d}s = 0$.

[21] Thus, the Hilbert transform (5) can be applied as

$$u_n(x)|_{x\in\Gamma_0} = \frac{1}{2\pi}\left[⨍_{\Gamma_0} \cot\frac{x-s}{2}\mathrm{d}u_0 + \int_{\Gamma_1} \cot\frac{x-s}{2}\gamma_0' u|_{\Gamma_1}\mathrm{d}s\right] \tag{9}$$

and

$$u^*_n(x)|_{x\in\Gamma_0} = \frac{1}{2\pi}\left[⨍_{\Gamma_0} \cot\frac{x-s}{2}\mathrm{d}u_0 + \int_{\Gamma_1} \cot\frac{x-s}{2}\gamma_0' u^*|_{\Gamma_1}\mathrm{d}s,\right] \tag{10}$$

where $\gamma_0' = \mathrm{d}/\mathrm{d}s\gamma_0$ is the derivative of linear operator γ_0. As $u_n|_{\Gamma_0} = u^*_n|_{\Gamma_0} = g_0$, by subtracting equation (9) with (10), we derive the result

$$\int_{\Gamma_1} \cot\frac{x-s}{2}\gamma_0'(u-u^*)|_{\Gamma_1}\mathrm{d}s = 0, x \in \Gamma_0. \tag{11}$$

It indicates that $u^* = u$, and from the Neumann trace theory, we can conclude that $u^*_{n1} = u_{n1}$.

[22] (2) The existence of an unique u_{t1} for solution of IBVP (7) can also be proved in a similar way.

[23] This completes the proof.

[24] *Corollary* 3.2
As Γ_0 contract to only one point on the enclosed circle, given the accurate data of (u_0, g_0), the IBVP (7) has still a unique solution of (u_{t1}, u_{n1}) on the inaccessible boundary of Γ_1.

3.2. New Solution Approach for the IBVPs

[25] First, we set $u_{t0} = \mathrm{d}u_0/\mathrm{d}s$ and $u_{n0} = g_0$, which are the known information on Γ_0, and set u_{t1}, u_{n1} as the pending solution of IBVP (7), and then we denote $u_t = u_{t0} \cup u_{t1}$ and $u_n = u_{n0} \cup u_{n1}$. Assumed that the constraint conditions of (3) and (4) are satisfied, the following equations can be derived with the Hilbert transform (5, 6), i.e.,

$$\int_{\Gamma_1} \cot\frac{x-s}{2}u_{t1}(s)\mathrm{d}s = f_n(x), x \in \Gamma_0 \tag{12}$$

and

$$\int_{\Gamma_1} \cot\frac{s-x}{2}u_{n1}(x)\mathrm{d}x = f_t(s), s \in \Gamma_0, \tag{13}$$

where $f_n(x) = 2\pi u_{n0}(x) - ⨍_{\Gamma_0} \cot\frac{x-s}{2}u_{t0}(s)\mathrm{d}s$ and $f_t(s) = 2\pi u_{t0}(s) - ⨍_{\Gamma_0} \cot\frac{s-x}{2}u_{n0}(x)\mathrm{d}x$ are the decided right item of equations (12) and (13), respectively. Both equations, (12) and (13), are integral equations of the first kind.

[26] Quadratures and the integral operators in equations (12) and (13) are discretized following the method of discrete vortices, which is reported to have high accuracy for singular integrals with Hilbert kernel [*Belotserkovsky and Lifanov*, 1993]. The circle is uniformly partitioned within the

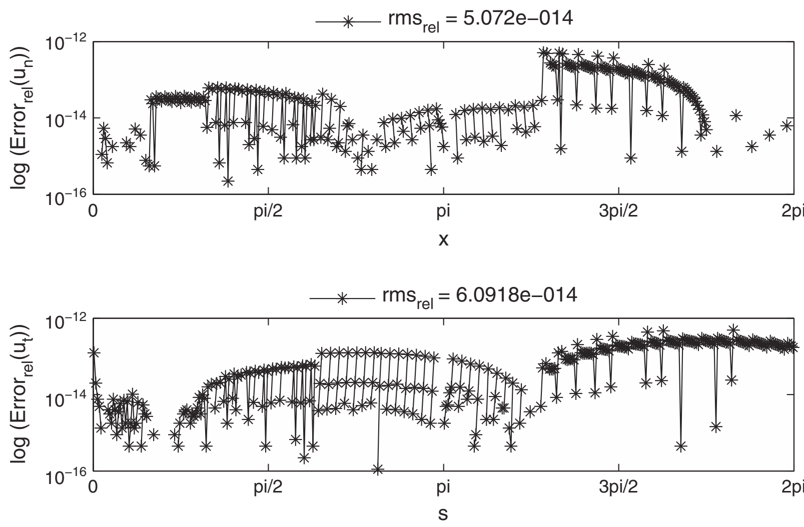

Figure 1. Numerical errors are plotted for benchmark testing of the quadrature scheme of Hilbert transform (5) on top panel and (6) on bottom panel, where the relative error $E_{rel} := \frac{\|\hat{u}_x - u_x\|}{\|u_x\|}$, $x = t, n$.

interval $[0, 2\pi]$ with a mesh size of $h = 2\pi/n$; the resultant operator equations are denoted in matrix form

$$\mathbf{A}\mathbf{u} = \mathbf{F}, \tag{14}$$

where $\mathbf{A}$ is the discrete integral operator, $\mathbf{u}$ is a column vector of the unknown quantities in equations (12) and (13), and $\mathbf{F}$ is another column vector calculated from the known quantities on Γ_0. Equation (14) may be severely ill-posed, if there is no enough information on Γ_0 or there are noises ($\mathbf{F} = \mathbf{F}^{\delta}$), and thus cannot be solved directly.

[27] We solve it through the Tikhonov regularization schemes [*Tikhonov and Arsenin*, 1977], which is formed by the weighted sum of a residual norm $C(\mathbf{u}) = \|\mathbf{A}\mathbf{u} - \mathbf{F}\|^2$ and a Sobolev norm $\Omega(\mathbf{u}) = \|\mathbf{L}\mathbf{u}\|^2$ ($\mathbf{L}$ is the discrete linear operator of the first derivatives and is selected for the need of smoothness). A regularized solution is found through minimizing this sum, i.e.,

$$\hat{\mathbf{u}} = \arg\min\{\alpha\|\mathbf{A}\mathbf{u} - \mathbf{F}\|^2 + \|\mathbf{L}\mathbf{u}\|^2\}. \tag{15}$$

This solution can be written in a matrix form as follows

$$\hat{\mathbf{u}}(\alpha) = (\alpha\mathbf{L}^T\mathbf{L} + \mathbf{A}^T\mathbf{A})^{-1}(\mathbf{A}^T\mathbf{F}), \tag{16}$$

where the upper index T denotes transposition and $\alpha > 0$, which is the only regularization parameter. We select the optimal value $\alpha = \alpha_{opt}$ through an iterative scheme (for details, see *Ramm* [2007])

$$\hat{\mathbf{u}}_{n+1} = e^{-h_n}\hat{\mathbf{u}}_n + (1 - e^{-h_n})(\alpha_n\mathbf{L}^T\mathbf{L} + \mathbf{A}^T\mathbf{A})^{-1}(\mathbf{A}^T\mathbf{F}), \tag{17}$$

where $\alpha(t) = \alpha_0/(1 + t)$, $h_n = t_{n+1} - t_n$, and $\alpha_n = \alpha(t_n)$, and we choose α_0 from the condition $\delta < \|\mathbf{A}\hat{\mathbf{u}}_{\alpha_0} - \mathbf{F}_{\delta}\| < 2\delta$, and

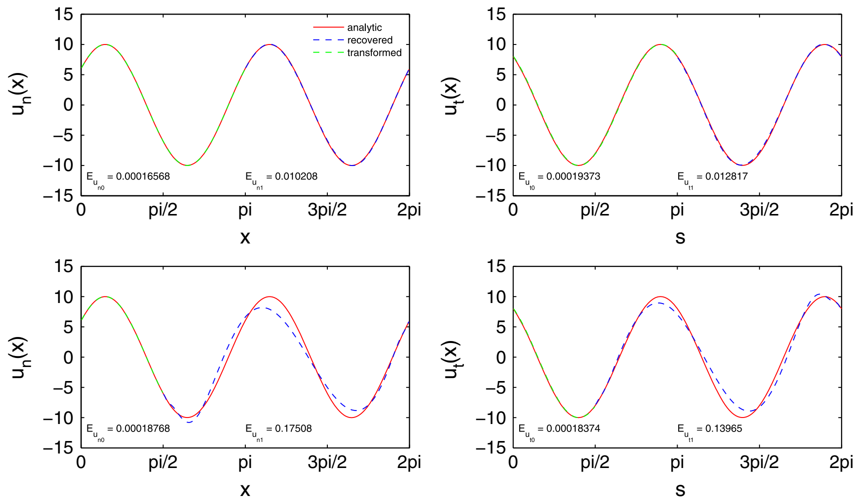

Figure 2. Benchmark testing results for IBVP solution, the recovered data on Γ_1 and the transformed data on Γ_0 are plotted against the analytical data, and top row for the case $\Gamma_0 = [0, \pi]$ and bottom row for case of $\Gamma_0 = [0, \pi/2]$. The error are defined by $\frac{\|\hat{u}_x - u_x\|}{\|u_x\|}$, $x = n1, t1, n0, t0$.

the iteration repeats until the condition $0.9\delta < \|\mathbf{A}\hat{\mathbf{u}}_n - \mathbf{F}_\delta\| < 1.001\delta$ is satisfied, where δ is the specified noise level.

4. Numerical Results

4.1. Benchmark Testing of the Hilbert Transform

[28] Quadrature of the Hilbert transform (5, 6) are discretized with the method of *Belotserkovsky and Lifanov* [1993]. A benchmark testing is carried out with the analytic function $u_t(s) = 2[\cos(s) + \sin(s)]$ and the exact dual function of $u_n(x) = 2[\cos(x) + \sin(x)]$. As shown in Figure 1, the RMS errors are of 10^{-14} order when the circle is partitioned by $n = 400$.

4.2. Benchmark Testing of the New Approach

[29] Integral equations (12, 13) are discretized in a similar way as in section (4.1). Another benchmark testing is carried out with the analytic function of $u_t(s) = -6\sin(2s)+8\cos(2s)$, and the exact dual function of $u_n(x) = 6\cos(2x) + 8\sin(2x)$. The known data are produced on Γ_0 through corresponding analytic functions. Recovered data on Γ_1 are produced by the new approach with case of $\Gamma_0 = [0, \pi]$ and $\Gamma_0 = [0, \pi/2]$, respectively. As shown by Figure 2, the relative error for the recovered data is at a level of about 1% for the case $\Gamma_0 = [0, \pi]$, i.e., the known information covered a half of the circle, while for the case of a quarter circle (bottom panel), it reaches the level of 13%–17%.

[30] As regards the transformed data on Γ_0 that are plotted on Figure 2, which is produced from the recovered data on Γ_1 plus corresponding known data on Γ_0 through the Hilbert transform described by equations (5) and (6), we find that the relative errors are all in the level of 0.01%, which indicates that the iterated Tikhonov regularization scheme is very effective in reducing the errors in transformed data.

[31] We can also find from the plots that bigger error in recovered data will result in bigger error in the transformed data, which indicates that further study is needed on setting of parameters, like h_n, α_0, and δ, and the ceasing condition in the iteration procedure of the regularization scheme. By careful controlling of the iteration procedure, we believe our new approach can produce a satisfied recovered results as Γ_0 becomes shorter and shorter, although for a situation setting by the Corollary 3.2, finding the stable solution of the IBVP (7) may be still a great challenge.

5. Summary and Conclusion

[32] The essential technique of GS reconstruction are investigated in this paper, which is formed into the IBVPs of the Laplace's equation on a circle. We find that there exists an Hilbert transform between the gradient components on the circle, which can be formed into the singular integrals with an Hilbert kernel. Existence and uniqueness of specified IBVPs are proved with a corollary from the theory of mixed boundary value problem for elliptic equations and the introduced Hilbert transform. New approach for solving the IBVP that is implemented through an iterated Tikhonov regularization scheme, is presented. Benchmark testings to the discrete schemes of the Hilbert transform and the new solution approach have shown the efficiency and robustness of the proposed method. The numerical results have also shown that there are room for further improvements of the new approach, as the error of transformed data (from IBVP solution) on Γ_0 is of order of 10^{-4}, but for transformed results from analytic data, it is of the order of 10^{-14}.

[33] In contrast to the prevailing approach for solving the IBVPs, we conclude that the new approach has the following triumphs: (1) It is developed based on a concise formula of the Hilbert transform, and it can be discretized by an accurate discrete vortex method, which is seldom been discussed in literatures to our knowledge. (2) With this new approach, we do not need to discretize the Laplace operator (e.g., the finite difference or the Galerkin method), or else it would result in numerical errors mixed with the ill-posedness of the suggested problem. (3) We need also not to prepare the first guess of the unknown data on the remaining boundary with this new approach; otherwise, it would result in impulses or discontinuities in its solution, which is in conflict with the $u \in L^2(\Omega)$ restriction. (4) We do not need to solve the boundary value problem in the iterated solution procedure, which shows the advantage of efficiency. The new approach is applicable to the reconstruction of two-dimensional coherent magnetic structures. New scheme for the IBVPs of GS equation and the Benchmark testing with its analytic solution will be presented in another publication.

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